AD-A046 711 NORTH CAROLINA STATE UNIV RALEIGH DEPT OF MECHANICAL--ETC F/G 20/4
THREE-DIMENSIONAL TURBULENT BOUNDARY LAYER ON A SPINNING CONE A--ETC(U)
OCT 77 C P FORD DAHC04-74-G-0206

UNCLASSIFIED ARO-12675.1-E NL

1 OF 2
AD
A046711

ARO 12675.1-E

12

AD A046711

Three-Dimensional Turbulent Boundary Layer on a Spinning Cone at Angle of Attack

by C. Phillip Ford III

OCT. 1977

Prepared by

Mechanical and Aerospace Engineering Department
North Carolina State University
Raleigh, North Carolina 27607

for

Engineering Sciences Division
U. S. Army Research Office
Post Office Box 12211
Research Triangle Park, N. C. 27709

(18) ARO (19) 12675.1-E

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER

2. GOVT ACCESSION NO.

3. RECIPIENT'S CATALOG NUMBER

(9)

4. TITLE (and Subtitle)

(6) THREE-DIMENSIONAL TURBULENT BOUNDARY LAYER ON A SPINNING CONE AT ANGLE OF ATTACK.

5. TYPE OF REPORT & PERIOD COVERED

Final rept.
1 Jun[redacted]74 [redacted]-30 Jun[redacted]77,

6. PERFORMING ORG. REPORT NUMBER

7. AUTHOR(s)

(10) C. Phillip Ford III

8. CONTRACT OR GRANT NUMBER(s)

(15) DAHC04-74-G-0206,
DAHC04-76-G-0004
DAAG29-76-G-0303

9. PERFORMING ORGANIZATION NAME AND ADDRESS

Mechanical and Aerospace Engineering Department
North Carolina State University
Raleigh, North Carolina 27607

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS

11. CONTROLLING OFFICE NAME AND ADDRESS

U. S. Army Research Office
Post Office Box 12211
Research Triangle Park, NC 27709

12. REPORT DATE

(11) Oct[redacted]1977

13. NUMBER OF PAGES

~~116~~ (12) 121p.

14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)

15. SECURITY CLASS. (of this report)

Unclassified

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

NA

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

NA

18. SUPPLEMENTARY NOTES

The findings in this report are not to be construed as an official Department of the Army position, unless so designated by other authorized documents.

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Three-Dimensional Turbulent Boundary Layer
Incompressible Flow
Momentum Integral Technique
Numerical Method - Implicit Finite Difference

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This report investigates the three-dimensional turbulent boundary layer on a spinning cone at small angles of attack in incompressible flow. It is assumed that the boundary layer has no effect on the inviscid flow and thus the inviscid flow is used as the outer boundary condition for the boundary layer equations. A momentum integral technique was used to reduce the governing equations to two. The two resulting partial differential equations were then solved by an implicit finite-difference technique of the marching type. Over

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

Unclassified

400714

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

20. ABSTRACT (continued)

The numerical method developed here gives results which follow the trend of available experimental data and other analytical results. This method was also found to be stable and accurate and fairly insensitive to the step size used in the finite-difference method. Computer solution could be found in the cases where spin rate produces surface velocity at the base of the cone less than the free stream velocity. Test cases were run with a free stream velocity of 60.96 meters per second with a cone half angle of 10 degrees. Angle of attack was either zero or five degrees with spin rates of 0, 400, and 500 revolutions per minute. Typical execution times was less than three minutes on an IBM 370/165 computer.

ACKNOWLEDGMENTS

This research was sponsored by the U. S. Army Research Office, Research Triangle Park, North Carolina 27709 under Grant Numbers DAHC04-74-G-0206, DAHC04-76-G-0004, and DAAG29-76-G-0303, covering the period from June 1, 1974 to June 30, 1977.

The author would like to express his appreciation to Dr. Robert E. Singleton of the U. S. Army Research Office for his review of the manuscript and the valuable assistance during all phases of this work. Dr. Singleton was also the project monitor for the Army grants under which this work was done.

The author would like to thank the following people at North Carolina State University:

1) Dr. Fred R. DeJarnette for his invaluable assistance especially in the writing of this paper.
2) Mrs. Deborah S. Clapp for typing the manuscript and assisting with the printing.

CONTENTS

1. INTRODUCTION

It is well known that spinning a cone at an angle of attack in flight will cause distortions to the cone's boundary layer. The asymmetrical boundary layer due to the combined spin and angle of attack of the cone produces an effective body shape which is asymmetric with respect to the plane of angle of attack. The inviscid flow over this effective body shape produces not only lift but a side force called the Magnus force or Magnus effect. This force is very important in the fact that it is an undamped force that acts on the body during its entire flight. Most studies of the Magnus effect have been restricted to either laminar boundary layers or supersonic turbulent boundary layers. This report investigates the distorted three-dimensional turbulent boundary layer on a spinning cone at small angles of attack in incompressible flow. The resulting boundary layer displacement shape could be used to determine an effective body shape and thus the Magnus force.

In 1972, White[1] developed a method to calculate coupled skin friction and heat transfer in two-dimensional turbulent boundary-layers and also for the calculation of three-dimensional turbulent skin-friction. In this study, White used a hodograph model to represent the crossflow velocity profile. In 1973, Jacobson, Vollmer, and Morton[2] studied velocity profiles of the incompressible laminar boundary layer on a spinning cone. They applied the 'Mangler' transformation to the cone coordinate system to develop the velocity profiles. These velocity profiles were then used to determine the displacement thickness and Magnus effect. Also, in 1973 White and Lessmann[3] introduced a paper on compressible turbulent

skin friction and heat transfer in three-dimensional boundary layers. In 1975, Sturek[4] introduced a paper on the effects of the three-dimensional boundary layer on the Magnus effect of spinning projectiles. In this study he found that the effective body shape is a major contributor to the Magnus effect, but the centrifugal pressure gradient is equally important. He also found that the shear stresses in the longitudinal and circumferential direction would affect the Magnus force.

During the period that the boundary layer was being investigated, the transition from laminar to turbulent boundary layers was also being studied. In 1972 Sturek[5] performed experimental studies of the boundary layer on a spinning cone at supersonic speeds. Besides investigating transition, these experiments also showed the effects of spin on the boundary layer of the cone. The boundary layer thickness and location of the boundary layer transition were determined, and the Magnus force was found to be very sensitive to the boundary layer shape. In 1973, Sturek[6] published a more detailed paper on the same subject. Also in 1973, Jacobson and Morton[7] analyzed the stability of the laminar boundary layer on a spinning cone. This study indicated that spin has a large effect on both laminar boundary layer stability and transition. Sturek[8], in 1974, published additional experimental results on boundary layer shapes and transition, and the results extended beyond spinning cones to spinning bodies of revolution. The bodies studied were sharp nosed and blunted cones with and without cylindrical afterbodies. Experimental results were obtained by Potter[9] in 1975 on the transition Reynolds numbers on axisymmetric bodies near the speed of sound.

This report concentrates on the incompressible turbulent boundary layer, while previous methods do not. In this report, the flow over the

cone is broken into two parts, the inviscid flow and the boundary layer. It is assumed that the boundary layer has no effect on the inviscid flow and thus the inviscid flow is used as one of the boundary conditions for the boundary layer equations. The other boundary conditions are on the surface of the cone itself. Although this report does not present results for the Magnus force, it could be determined by using the results from the boundary layer calculations to determine an effective body shape and then perform a new inviscid flow field calculation.

This report is divided into five major sections after the introduction. The first section describes the basic equations and the development of the velocity profiles used to apply the momentum integral technique. Then these velocity profiles are applied to the boundary layer equations through the momentum integral technique to develop two governing, quasi-linear, partial differential equations. The next section describes the method for finding the location of transition from a laminar to a turbulent boundary layer and the initial turbulent conditions on the transition line. The third section describes the numerical technique used to integrate the governing equations. Section four describes the results that are obtained from the computer analysis while section five gives conclusions derived from these results.

SYMBOLS

a	parameter defined by equation (A-49) in Appendix A
b	parameter defined by equation (A-50) in Appendix A
A,B,C,D,E	coefficients in equations (36) and (37)
A_3	parameter defined by equation (A-3) in Appendix A
A_4	parameter defined by equation (A-7) in Appendix A
B_3	parameter defined by equation (A-6) in Appendix A
B_4	parameter defined by equation (A-13) in Appendix A
$\overline{C}$	vector defined by equation (41)
C_o	parameter of integration in equation (23)
c([illegible]	element of $\overline{c}$
[illegible]	coefficient of skin friction
$\overline{F}$	solution vector of equation (41)
G_1-G_{21}	coefficients defined in A
H	coefficient matrix of equation (41)
h(i,j)	element of matrix H
i	used as matrix subscript
I_0-I_2	coefficients defined in Appendix A
IC	parameter used to describe matrix equation
IM	parameter used to describe matrix equation
IP	parameter used to describe matrix equation
j	used as matrix subscript
J_1-J_6	coefficients defined in Appendix A
k	subscript used in equation (40)
L	length of cone, m

m	exponent used in equation (35)
M	number of grid points in ϕ-direction
N	coefficient defined in equation (4)
P	pressure, N/m^2
P_t	empirical parameter for determining effects of angle of attack on transition
q	velocity parameter defined in equation (21)
q_o	value of q where $y^+ = y_o^+$ and U^+ approaches zero
R_L	Reynolds number based on the length of cone and free stream properties defined in equation (A-47)
s	dimensional distance along cone generator, m
u	velocity component in s-direction, m/s
u_∞	free stream velocity, m/s
u^*	skin friction velocity parameter as defined by equation (13), m/s
u^+	nondimensional velocity defined by equation (14)
U	nondimensional velocity in S direction defined by $U = u/u_\infty$
v	velocity component in the y-direction, m/s
w	velocity component in the ϕ-direction, m/s
w^+	nondimensional velocity parameter defined by equation (15)
W	nondimensional velocity in the ϕ-direction defined by $W = w/u_\infty$
x	nondimensional distance along cone generator defined by $x = s/L$
x_t	nondimensional position of the transition line defined by equation (35)
y	height normal to cone surface, m
y^+	nondimensional height as defined by equation (16)
y_o^+	nondimensional height where $q = q_o$

Z	empirical parameter of equation (35)
α	angle of attack, degrees
α_s	parameter defined by equation (18)
β	parameter defined by equation (30)
δ	boundary layer thickness, m
δ^+	nondimensional boundary layer thickness
ε	parameter defined by equation (A-48) in Appendix A
Θ	tangent of angle between shear in ϕ-direction and shear in the s direction defined by equation (27)
κ	Von Karmen's constant used in Prandtl mixing length equals 0.4
λ	skin friction parameter defined by $\lambda = \sqrt{2/c_f}$ or $\lambda = u_e^+$
μ	coefficient of viscosity in kg/m-sec (1.798×10^{-5} kg/m-sec)
ν	kinematic viscosity in m^2/sec (1.4639×10^{-5} m^2/sec)
ξ	arbitrary parameter used in equations (38) and (39)
ρ	density in kg/m^3 (1.2283 kg/m^3)
σ	parameter defined by equation (29)
τ_s	shear parameter in s direction N/m^2
τ_ϕ	shear parameter in ϕ-direction N/m^2
ϕ	coordinate measured around cone, degrees
Φ_p	velocity potential defined by equation (1)
ψ	cone semi-vertex angle, degrees
ω	spin rate, revolutions per minute

SUBSCRIPTS

e	in external flow (external to boundary layer)
lam	laminar boundary layer property

i	denotes grid point in ϕ-direction
turb	turbulent boundary layer property
w	property at the wall or surface of cone
δ^+	property at edge of boundary layer
∞	free stream property

SUPERSCRIPT

+	denotes nondimensional quantity
n	denotes grid point in X-direction

2. BASIC EQUATIONS

2.1 Inviscid Flow Theory

The inviscid flow theory used in this study was developed by Jacobson, Vollmer, and Morton[2]. The coordinate system is shown in Figure 1. The coordinate s is measured along the generator of the cone surface, y is measured normal to the surface, and ϕ is the circumferential angle. The semi-vertex angle of the cone is ψ, and the angle of attack is α. The velocity components u, v, and w are in the s, y, and ϕ directions, respectively. The angular velocity of the cone, ω, is assumed to be constant, and it is positive in the direction of decreasing ϕ.

Basically the incompressible inviscid flow is described by a velocity potential, ϕ_p, which satisfies Laplace's equation

$$\nabla^2 \phi_p = 0 \quad . \tag{1}$$

The boundary conditions on the cone require the velocity to be tangent to the surface, hence

$$v = 0 \quad \text{at} \quad y = 0 \quad ,$$

at infinity the flow is undisturbed

$$\nabla\phi_p = \vec{u}_\infty \quad \text{at} \quad y = \infty \ .$$

Since equation (1) is linear, the solution can be expressed as the superposition of two velocity potentials, one due to the axial flow and the other due to the cross flow. To simplify the solution, Jacobson, et al., assumed a slender cone at small angles of attack. They also assumed that the boundary layer did not effect the external flow. The flow around the body was approximated by a distribution of sources and sinks along the axis of the body. This analysis leads to the following

velocity components at the cone's surface

$$\frac{u_e}{u_\infty} = \left(\frac{s}{L}\right)^N - 2\alpha \sin\psi \cos\phi \tag{2}$$

$$v_e = 0$$

$$\frac{w_e}{u_\infty} = 2\alpha \sin\phi \tag{3}$$

where N is given by the solution to

$$\cos\left[\left(N + \frac{1}{2}\right)(\pi - \psi) - \frac{\pi}{4}\right] + \cos(\psi)\cos\left[\left(N + \frac{3}{2}\right)(\pi - \psi) - \frac{\pi}{4}\right] = 0 \; . \tag{4}$$

Euler's equations can now be used to calculate the pressure gradients as follows

$$-\frac{1}{\rho}\frac{\partial P}{\partial s} = u_e \frac{\partial u_e}{\partial s} + \frac{w_e}{s \sin\psi}\frac{\partial u_e}{\partial \phi} - \frac{w_e^2}{s} \tag{5}$$

and

$$-\frac{1}{\rho\, s \sin\psi}\frac{\partial P}{\partial \phi} = u_e \frac{\partial w_e}{\partial s} + \frac{w_e}{s \sin\psi}\frac{\partial w_e}{\partial \phi} + \frac{u_e w_e}{s} \tag{6}$$

Thus, the velocity components and pressure gradients are known at any position external to the boundary layer.

2.2 Turbulent Boundary Layer Equations

The boundary layer equations for the coordinate system described above are[10]

<u>Continuity</u>

$$\frac{1}{s \sin\psi}\frac{\partial}{\partial s}(u\, s \sin\psi) + \frac{1}{s \sin\psi}\frac{\partial w}{\partial \phi} + \frac{\partial v}{\partial y} = 0 \tag{7}$$

s Momentum

$$u \frac{\partial u}{\partial s} + \frac{w}{s \sin \psi} \frac{\partial u}{\partial \phi} + v \frac{\partial u}{\partial y} - \frac{w^2}{s} = -\frac{1}{\rho} \frac{\partial P}{\partial s} + \frac{1}{\rho} \frac{\partial \tau_s}{\partial y} \tag{8}$$

ϕ Momentum

$$u \frac{\partial w}{\partial s} + \frac{w}{s \sin \psi} \frac{\partial w}{\partial \phi} + v \frac{\partial w}{\partial y} + \frac{uw}{s} = - \frac{1}{\rho\, s \sin \psi} \frac{\partial P}{\partial \phi} + \frac{1}{\rho} \frac{\partial \tau_\phi}{\partial y} \tag{9}$$

and y Momentum

$$\frac{\partial P}{\partial y} = 0 \quad . \tag{10}$$

In laminar flow, the values of τ_s and τ_ϕ are

$$\tau_s = \mu \frac{\partial u}{\partial y}$$

and

$$\tau_\phi = \mu \frac{\partial w}{\partial y} \quad ,$$

while their values in turbulent flow are given in the next section. With these boundary layer equations are two sets of boundary conditions. On the surface of the cone the no-slip boundary conditions gives

$$y = 0$$

$$u = u_w = 0$$

$$v = v_w = 0$$

$$w = w_w = - \omega s \sin \psi \quad . \tag{11}$$

The other set of boundary conditions is at the edge of the boundary layer where the velocities must match the external flow. Thus, at

$$y = \delta$$

$$u = u_e$$

$$v = v_e = 0$$

$$w = w_e$$

where u_e is given by equation (1) and w_e is given by equation (2).

2.3 Turbulent Boundary Layer Velocity Profiles

To apply a momentum integral technique to the boundary layer equations, choices for the u and w velocity profiles across the boundary layer need to be made. First the u-velocity profile will be derived and then the w-velocity profile.

Near the wall, the s momentum equation, equation (8), becomes

$$-\frac{w_w^2}{s} = -\frac{1}{\rho}\frac{\partial P}{\partial s} + \frac{1}{\rho}\frac{\partial \tau_s}{\partial y}$$

which is exact at the wall and approximate away from the wall. Now assuming that this equation holds throughout the boundary layer, integrate it with respect to y to obtain the following

$$\tau_s = \left[\frac{\partial P}{\partial s} - \rho\frac{w_w^2}{s}\right] y + (\tau_s)_w \quad .$$

Now substitute, in the above equation for τ_s, a Prandtl mixing length, κy where $\kappa = 0.4$, of the form

$$\tau_s = \rho\kappa^2 y^2 \left|\frac{\partial u}{\partial y}\right| \frac{\partial u}{\partial y}$$

to obtain the following

$$\frac{\partial u}{\partial y} = \pm\frac{1}{\kappa y}\left[\frac{(\tau_s)_w}{\rho} + \left(\frac{1}{\rho}\frac{\partial P}{\partial s} - \frac{w_w^2}{s}\right) y\right]^{1/2} \quad . \qquad (12)$$

The negative sign in the above equation denotes separated flow because of the negative velocity gradient; the plus sign denotes attached flow. Define u* by

$$u^* = \left(\frac{(\tau_s)_w}{\rho}\right)^{1/2} \tag{13}$$

and note that u* has the dimensions of speed. Then define the following dimensionless parameters:

$$u^+ = u/u^* \tag{14}$$

$$w^+ = w/u^* \tag{15}$$

$$y^+ = \frac{u^* y}{\nu_w} \quad . \tag{16}$$

Substitute these parameters into equation (12) with the plus sign to obtain

$$\frac{\partial u^+}{\partial y^+} = \frac{1}{\kappa y^+}\left[1 + \left(\frac{\nu_w}{u^* (\tau_s)_w}\frac{\partial P}{\partial s} - \frac{w_w^2\ \nu_w}{(u^*)^2\ s\ u^*}\right)y^+\right]^{1/2} \tag{17}$$

Now define part of the square root term in the above equation by

$$\alpha_s = \frac{\nu_w}{u^* (\tau_s)_w}\frac{\partial P}{\partial s} - \frac{w_w^2\ \nu_w}{s\ (u^*)^3} = \frac{\nu_w}{(u^*)^3}\left[\frac{1}{\rho}\frac{\partial P}{\partial s} - \frac{w_w^2}{s}\right] \quad . \tag{18}$$

Substitute for $\frac{1}{\rho}\frac{\partial P}{\partial s}$ from equation (5) and for w_w from equation (11) into the above equation to obtain

$$\alpha_s = \frac{\nu_w}{(u^*)^3}\left[- u_e \frac{\partial u_e}{\partial s} + \frac{w_e}{s}\left(w_e - \frac{1}{\sin\psi}\frac{\partial u_e}{\partial \phi}\right) - \omega^2\ s\ \sin^2\psi\right] \quad . \tag{19}$$

Use the definition of α_s from equation (18) to get equation (17) in the following form

$$\frac{\partial u^+}{\partial y^+} = \frac{1}{\kappa y^+}\left(1 + \alpha_s\ y^+\right)^{1/2} \quad . \tag{20}$$

Now define q by

$$q = \left(1 + \alpha_s\ y^+\right)^{1/2} \tag{21}$$

and then rearrange it to get

$$y^+ = \frac{1}{\alpha_s}(q^2 - 1)$$

and then differentiate this equation to obtain

$$dy^+ = \frac{1}{\alpha_s} 2q\, dq \quad . \tag{22}$$

Replace y^+ in terms of q in equation (20) and integrate, holding s and ϕ constant, to obtain

$$u^+ = \frac{1}{\kappa}\{2q - \ln \frac{q+1}{q-1}\} + C_o \tag{23}$$

where C_o is a parameter of integration and may be a function of s and ϕ. Let q_o be the value of q as u^+ approaches zero from above. Then equation (23) gives

$$u^+ = \frac{1}{\kappa}\left\{2(q - q_o) + \ln\left[\frac{q - 1}{q + 1}\left(\frac{q_o + 1}{q_o - 1}\right)\right]\right\} \quad . \tag{24}$$

Now using equation (21) the following limit may be obtained

$$\lim_{y^+ \to y_o^+}\left(\frac{q_o + 1}{q_o - 1}\right) = \frac{1 + \frac{1}{2}\alpha_s\, y_o^+ + \mathcal{O}(y_o^{+2}) + 1}{1 + \frac{1}{2}\alpha_s\, y_o^+ + \mathcal{O}(y_o^{+2}) - 1}$$

$$= \frac{2 + \frac{1}{2}\alpha_s\, y_o^+ + \mathcal{O}(y_o^{+2})}{\frac{1}{2}\alpha_s\, y_o^+ + \mathcal{O}(y_o^{+2})} = \frac{4}{\alpha_s\, y_o^+} + \mathcal{O}(1)$$

where y_o^+ is the height where $q = q_o$ and also if y_o^+ is small in comparison to δ, then

$$(q - q_o) = q - \left(1 + \frac{1}{2}\alpha_s\, y_o^+ + \mathcal{O}(y_o^{+2})\right) \simeq (q - 1) \quad ,$$

so that equation (24) becomes approximately

$$u^+ = \frac{1}{\kappa}\left\{2(q - 1) + \ln\left[\frac{q - 1}{q + 1}\,\frac{4}{\alpha_s\, y_o^+}\right]\right\} . \tag{25}$$

Equation (25) is the velocity profile used for the u-velocity. When equation (25) is applied at the edge of the boundary layer it gives

$$u_e^+ = \frac{1}{\kappa}\left\{2(q_{\delta^+} - 1) + \ell n\left[\frac{q_{\delta^+} - 1}{q_{\delta^+} + 1}\left(\frac{4}{\alpha_s\ y_o^+}\right)\right]\right\} \tag{26}$$

where q_{δ^+} is the value of q at the edge of the boundary layer.

Now that a [illegible] u-velocity profile has been found, a velocity profile fo[illegible]verse velocity, w, is needed. First define the parameter Θ as follows

$$\Theta = \frac{\tau_\phi}{\tau_s} \quad . \tag{27}$$

An analysis of the w-velocity, as done with the u-velocity, produced a velocity correlation that was much too complicated and thus, some other means was sought. A hodograph model[1] is used, in this report, of the form

$$\frac{w - w_w}{u} = \Theta\left(1 - \frac{y}{\delta}\right)^2 + f\left(\frac{y}{\delta}\right)$$

where

$$w \to w_e \quad \text{as} \quad y \to \delta$$

and

$$w \to w_w \quad \text{as} \quad y \to 0 \quad .$$

To satisfy the above boundary conditions a hodograph model of the following form was used

$$\frac{w - w_w}{u} = \Theta\left(1 - \frac{y}{\delta}\right)^2 + \left(\frac{w_e}{u_e} - \frac{w_w}{u_e}\right)\frac{y}{\delta}\left(2 - \frac{y}{\delta}\right) \tag{28}$$

Now define σ and β as

$$\sigma = \frac{w_w}{u_e} = -\ \frac{\omega\ s\ \sin\psi}{u_e} \tag{29}$$

and

$$\beta = \frac{w_e}{u_e} \tag{30}$$

and substitute them into equation (28) to get

$$w^+ = u_e^+ \sigma + u^+ \left[\Theta \left(1 - \frac{y^+}{\delta^+}\right)^2 + (\beta - \sigma) \frac{y^+}{\delta^+} \left(2 - \frac{y^+}{\delta^+}\right) \right] \tag{31}$$

This is the cross flow model used. Figures 2 through 7 show the u-velocity profile while Figures 8 through 11 depict the w-velocity profile.

2.4 The Governing Equations

This section describes the procedure used to derive the governing equations. A complete derivation of the governing equations is given in Appendix A.

The first step is to take partial derivatives of the u-velocity profile (equation (25)) and the w-velocity profile (equation (31)) with respect to s and ϕ. Then substitute these derivatives into the continuity equation (equation (7)) and then integrate with respect to y to obtain the following form

$$\begin{aligned} -\frac{v}{\nu_w} = {} & A_3 y^+ u^+ + \frac{1}{s} I_0 + \left\{ \frac{3\alpha_s}{u_e^+} \frac{\partial u_e^+}{\partial s} + \nu_w (u_e^+)^3 \left[\frac{\partial^2 \left(\frac{1}{u_e}\right)}{\partial s^2} + A_4 \right] \right\} I_2 \\ & + \frac{1}{s \sin \psi} \left\{ B_3 y^+ w^+ + \left[\frac{3\alpha_s}{u_e^+} \frac{\partial u_e^+}{\partial \phi} + \nu_w (u_e^+)^3 \left(\frac{\partial^2 \left(\frac{1}{u_e}\right)}{\partial s \, \partial \phi} + B_4 \right) \right] \right. \\ & \left[J_2 - \frac{\delta^+ (q_{\delta^+} - 1)}{\alpha_s q_{\delta^+}} J_3 \right] + \frac{\partial u_e^+}{\partial \phi} \frac{\kappa \delta^+}{q_{\delta^+}} J_3 + \frac{\partial \Theta}{\partial \phi} J_4 + \frac{\partial \beta}{\partial \phi} J_5 + \frac{\partial \sigma}{\partial \phi} J_6 \\ & \left. + \frac{\partial u_e^+}{\partial \phi} \sigma y^+ \right\} . \end{aligned} \tag{32}$$

Next, derivatives of the velocity profiles and equation (32) are substituted into the s-momentum equation (equation (8)), integrated with respect to y, and non-dimensionalized to obtain

$$\left(G_1 - 3\,\alpha_s\,G_2\right)\frac{\partial\lambda}{\partial x} + \frac{1}{x\sin\psi}\left[G_3 - 3\,\alpha_s\,G_4 + \lambda\,G_5 + \sigma\,\lambda\left(\delta^+\,\lambda - I_0^{(e)}\right)\right]\frac{\partial\lambda}{\partial\phi}$$

$$+ \frac{\lambda\,G_6}{x\sin\psi}\frac{\partial\Theta}{\partial\phi} + \frac{\lambda}{U_e}\frac{\partial U_e}{\partial x}\left(\lambda^2\,\delta^+ - G_1\right) + \frac{\lambda}{U_e}\frac{\partial U_e}{\partial\phi}\frac{1}{x\sin\psi}$$

$$\left(\beta\,\lambda^2\,\delta^+ - G_3 - \beta\,G_7 - \sigma\,G_8\right) + \frac{\lambda}{x\sin\psi}\frac{\beta}{W_e}\frac{\partial W_e}{\partial\phi}G_7$$

$$- \frac{\lambda^4}{R_L}\left\{G_2\left[\frac{\partial^2\left(\frac{1}{U_e}\right)}{\partial x^2} + a\right] + \frac{G_4}{x\sin\psi}\left[\frac{\partial^2\left(\frac{1}{U_e}\right)}{\partial x\,\partial\phi} + b\right]\right\}$$

$$+ \frac{\lambda}{x}\left(G_9 - \beta^2\,\delta^+\,\lambda^2\right) = U_e\,R_L \quad . \tag{33}$$

Similarily, derivatives of the velocity profiles and equation (32) are substituted into the ϕ-momentum equation (equation (9)), integrated with respect to y, and non-dimensionalized to obtain

$$\left(G_3 - 3\,\alpha_s\,G_{10} - \lambda\,G_{11} - \sigma\,\lambda\,I_0^{(e)}\right)\frac{\partial\lambda}{\partial x} + \frac{1}{x\sin\psi}\left[G_{12} - 3\,\alpha_s\,G_{13}\right.$$

$$\left. - \lambda\,G_{14} - \sigma\,\lambda\left(2\,J_0^{(e)} - \delta^+\,\beta\,\lambda\right)\right]\frac{\partial\lambda}{\partial\phi} - G_{15}\,\lambda\,\frac{\partial\Theta}{\partial x} - \frac{G_{16}}{x\sin\psi}\,\lambda\,\frac{\partial\Theta}{\partial\phi}$$

$$+ \frac{\lambda}{U_e}\frac{\partial U_e}{\partial x}\left(\sigma\,G_{18} + \beta\,G_{20} - G_3\right) + \frac{\lambda}{U_e}\frac{\partial U_e}{\partial\phi}\frac{1}{x\sin\psi}\left(\sigma\,G_{19} + \beta\,G_{21} - G_{12}\right)$$

$$+ \frac{\lambda}{W_e}\frac{\partial W_e}{\partial x}\,\beta\left(\lambda^2\,\delta^+ - G_{20}\right) + \frac{\lambda}{x\sin\psi}\frac{\beta}{W_e}\frac{\partial W_e}{\partial\phi}\left(\beta\,\lambda^2\,\delta^+ - G_{21}\right)$$

$$- \frac{\lambda^4}{R_L}\left\{G_{10}\left[\frac{\partial^2\left(\frac{1}{U_e}\right)}{\partial x^2} + a\right] + \frac{G_{13}}{x\sin\psi}\left[\frac{\partial^2\left(\frac{1}{U_e}\right)}{\partial x\,\partial\phi} + b\right]\right\} + \frac{\lambda}{x}\left(\beta\,\lambda^2\,\delta^+\right.$$

$$\left. - G_{17} - \sigma\,G_{18}\right) = U_e\,R_L\,\Theta \quad . \tag{34}$$

Equations (33) and (34) are the governing equations, where x and ϕ are the independent variables while λ and Θ are the dependent variables and λ is defined by

$$\lambda = \sqrt{\frac{2}{c_f}}$$

where c_f is the turbulent skin friction coefficient in the s direction. These equations, equations (33) and (34), describe the turbulent boundary layer on a spinning cone at small angle of attack assuming that the velocity profiles given in equation (25) and equation (31) are valid.

3. TRANSITION FROM LAMINAR TO TURBULENT BOUNDARY LAYER

Before a finite difference method can be applied to the governing equations, a starting line with known quantities is necessary. The flow over the cone was assumed to be laminar from the nose to the transition line and then abruptly becomes turbulent thereafter with no transition region. The laminar boundary layer equations over the cone are solved by the method of Jacobson, Vollmer, and Morton[2]. Since equation (33) and equation (34) are valid only in the turbulent flow, the transition line was chosen for the starting line.

This section is divided into two subsections. The first describes the position and shape of the transition line, and the second discusses the method used for determining the values of turbulent parameters on the starting line.

3.1 Transition

Three effects dealing with transition were investigated. First is where the location of transition occurs on a cone at zero angle of attack without spin. The second is the effect of angle of attack on the location of transition, and thirdly, is the distortion caused by spin.

The first effect has been determined empirically from experimental data by Potter[9] for the special case of zero angle of attack without spin. However, his equation for the transition location was for Reynolds numbers on the order of 10^6. His Reynolds number was based on the external flow parameters at the transition line and the distance the flow has traveled over the cone to the transition line. Since the cases considered here are for Reynolds numbers less than 10^4, Potter's experimental

data was used to derive the following empirical relation for the transition position, x_t

$$x_t = Z\left(\frac{u_e}{\nu_w}\right)^m \tag{35}$$

where $Z = 883.25$ and $m = -0.33452$. Since u_e itself is a function of x, equation (35) must be solved iteratively for x_t. This value of x_t determines the location of the transition line, which on a cone at zero angle of attack without spin is circular.

The effect of angle of attack without spin on the transition line has also been estimated empirically. This empirical method was guided using the experimental results found by Sturek[5,6]. The basis for this method is two ellipses joined together as depicted in Figure 12. This figure is a view looking along the axis of revolution of the cone and the distance from the center to the edge of the ellipse is the distance from the nose of the cone to transition along a cone generator. These two ellipses have one common axis of length x_t, this is the major axis of the ellipse on the leeward side and the minor axis of the ellipse on the windward side. The major axis of the ellipse on the windward side is $(1 + P_t\,\alpha)x_t$ while the minor axis of the other ellipse is $(1 - P_t\,\alpha)x_t$. The parameter P_t was empirically found to be 5.73.

The effect of spin was added by a point by point rotation of the above described transition line. The amount of rotation caused by spin is found by a two step process. First the location of maximum laminar momentum thickness is found for a cone at angle of attack without spin. Next the location of maximum laminar momentum thickness was found for the cone at angle of attack with spin. The angular difference in the location

of these momentum thicknesses is the amount the transition line is rotated about the cones axis of revolution.

This method of describing the transition line on the cone matched the experimental data very well[5,6,9]. Now that the transition line is known, the laminar solution can be applied up to this line. Then the laminar skin friction coefficient, c_f, and Θ are known at the transition line. These values are used in the next section to estimate the corresponding turbulent values of skin friction coefficient and Θ.

3.2 Turbulent Flow Properties on the Transition Line

Now that laminar boundary-layer properties are known on the transition line, they can be transformed to turbulent properties. The necessary turbulent data are the skin friction coefficient in the s-direction, c_f, and the shear angle, Θ. Because of a lack of detailed information, it is assumed that the skin friction in the s-direction changes across transition, while skin friction in the ϕ-direction remains the same across transition and hence Θ must change across transition. Because of the assumed instantaneous transition from a laminar to a turbulent boundary layer, there must be an instantaneous jump from laminar to turbulent values of c_f and Θ.

After a study of experimental and analytical work and shapes of boundary layers on different bodies, it was noticed that the boundary layer thickness, δ, is continuous across transition while its derivative with respect to s, $\frac{\partial \delta}{\partial s}$, is not continuous. This condition was assumed to hold true for the instantaneous transition. Since the laminar solution is known up to transition, the laminar boundary layer thickness at transition can be found. Then a skin friction jump will be found which makes the turbulent boundary

of these momentum thicknesses is the amount the transition line is rotated about the cones axis of revolution.

This method of describing the transition line on the cone matched the experimental data very well[5,6,9]. Now that the transition line is known, the laminar solution can be applied up to this line. Then the laminar skin friction coefficient, c_f, and Θ are known at the transition line. These values are used in the next section to estimate the corresponding turbulent values of skin friction coefficient and Θ.

3.2 Turbulent Flow Properties on the Transition Line

Now that laminar boundary-layer properties are known on the transition line, they can be transformed to turbulent properties. The necessary turbulent data are the skin friction coefficient in the s-direction, c_f, and the shear angle, Θ. Because of a lack of detailed information, it is assumed that the skin friction in the s-direction changes across transition, while skin friction in the ϕ-direction remains the same across transition and hence Θ must change across transition. Because of the assumed instantaneous transition from a laminar to a turbulent boundary layer, there must be an instantaneous jump from laminar to turbulent values of c_f and Θ.

After a study of experimental and analytical work and shapes of boundary layers on different bodies, it was noticed that the boundary layer thickness, δ, is continuous across transition while its derivative with respect to s, $\frac{\partial \delta}{\partial s}$, is not continuous. This condition was assumed to hold true for the instantaneous transition. Since the laminar solution is known up to transition, the laminar boundary layer thickness at transition can be found. Then a skin friction jump will be found which makes the turbulent boundary

layer thickness at the transition line equal to the laminar boundary layer thickness.

The laminar boundary layer thickness at transition is defined as the height, y, where $u/u_e = 0.99$. Then equation (26) is used to determine the skin friction jump in the following manner. From the inviscid flow solution, u_e^+ is known but q_{δ^+}, α_s, and y_0^+ are not known. Notice that α_s is a function of s, ϕ, and c_f, while q_{δ^+} is a function of δ and α_s. At this point the only unknowns in equation (26) are c_f and y_0^+. As it turns out, y_0^+ is a free parameter. The parameter y_0^+ is in equation (26) due to the fact that the parameter of integration, C_0 in equation (23) could not be evaluated at y = 0 because the log term would approach infinity. The value of y_0^+ was chosen so that equation (26) yielded a realistic c_f. The parameter y_0^+ could have been determined uniquely by first calculating the turbulent c_f by the method described by Spalding and Chi[11], which relates turbulent skin friction to properties at the edge of the boundary layer, and then solving equation (26) for y_0^+.

layer thickness at the transition line equal to the laminar boundary layer thickness.

The laminar boundary layer thickness at transition is defined as the height, y, where $u/u_e = 0.99$. Then equation (26) is used to determine the skin friction jump in the following manner. From the inviscid flow solution, u_e^+ is known but q_{δ^+}, α_s, and y_o^+ are not known. Notice that α_s is a function of s, ϕ, and c_f, while q_{δ^+} is a function of δ and α_s. At this point the only unknowns in equation (26) are c_f and y_o^+. As it turns out, y_o^+ is a free parameter. The parameter y_o^+ is in equation (26) due to the fact that the parameter of integration, C_o in equation (23) could not be evaluated at y = 0 because the log term would approach infinity. The value of y_o^+ was chosen so that equation (26) yielded a realistic c_f. The parameter y_o^+ could have been determined uniquely by first calculating the turbulent c_f by the method described by Spalding and Chi[11], which relates turbulent skin friction to properties at the edge of the boundary layer, and then solving equation (26) for y_o^+.

4. NUMERICAL METHOD

Generally an analytical solution to the governing equations of a cone with spin and angle of attack cannot be obtained. Therefore, a numerical method is needed to solve them on a digital computer. The numerical method is dependent on the coefficients in the governing equations (equations (33) and (34)) because they determine if the equations are elliptic, parabolic, or hyperbolic. The full development of these coefficients is shown in Appendices B and C. Let it suffice here to say that all coefficients in the governing equations can be described as functions of the independent variables, the dependent variables, and ω.

Equations (33) and (34) can be written in the following form

$$A_1 \frac{\partial \lambda}{\partial x} + B_1 \frac{\partial \lambda}{\partial \phi} + C_1 \frac{\partial \Theta}{\partial x} + D_1 \frac{\partial \Theta}{\partial \phi} + E_1 = 0 \tag{36}$$

$$A_2 \frac{\partial \lambda}{\partial x} + B_2 \frac{\partial \lambda}{\partial \phi} + C_2 \frac{\partial \Theta}{\partial x} + D_2 \frac{\partial \Theta}{\partial \phi} + E_2 = 0 \; . \tag{37}$$

Note that the coefficients of the derivatives in the above equations are functions of the dependent and independent variables and, thus these equations are quasi-linear. In addition, an analysis of the coefficients showed that the equations are hyperbolic. Hence, the solution can be started from an initial data line (the transition line here) and marched downstream.

Inherent to any finite-difference method are the questions of stability and accuracy. Also since the governing equations are hyperbolic, care must be taken to not exceed the zone of influence of the initial data line segments (mesh width). The characteristic lines of the governing equations were found to be practically parallel to the cone generators such that they

crossed only at large distances downstream. Thus, the step sizes were limited by stability and accuracy only. Explicit finite-difference methods were tried first, but all were found to require unrealistically small step sizes to maintain stability. Thus, an implicit finite-difference scheme was decided upon.

Now that Θ and c_f are known on the starting line (see section 3), λ is also known on the starting line since

$$\lambda = \sqrt{\frac{2}{c_f}}$$

This information is sufficient to start a finite-difference solution of equation (33) and (34). The following description of the finite-difference method used is described for a circular starting line. However, the technique could be modified along with a change of axis system so that it could be applied to the skewed transition line which exists on a cone with spin and angle of attack.

First a two-dimensional grid is placed over the x-ϕ coordinates as shown in Figure 13. The superscript 'n' denotes the x-position such that $x^{n+1} = x^n + \Delta x$, while the subscript 'i' denotes the ϕ-position such that $\phi_{i+1} = \phi_i + \Delta\phi$. Assuming that the data is known on the circle where $x = x^n$ and the unknown data is on the next line, $x = x^{n+1}$, then the derivatives of some variable, say ξ, are approximated by difference quotients in the x-direction (marching direction) by a backward difference quotient

$$\left.\frac{\partial\xi}{\partial x}\right|_i^{n+1} \simeq \frac{\xi_i^{n+1} - \xi_i^n}{\Delta x} \tag{38}$$

and in the ϕ-direction by a central difference quotient

$$\left.\frac{\partial \xi}{\partial \phi}\right|_{i}^{n+1} \simeq \frac{\xi_{i+1}^{n+1} - \xi_{i-1}^{n+1}}{2\,\Delta\phi} \tag{39}$$

The first difference quotient is a first order approximation to the derivative. These estimates of the derivatives are applied to equations (36) and (37) and rearranged to obtain

$$\begin{aligned} &- \frac{B_{k_i}^{n+1}}{2\Delta\phi}\,\lambda_{i-1}^{n+1} + \frac{A_{k_i}^{n+1}}{\Delta x}\,\lambda_{i}^{n+1} + \frac{B_{k_i}^{n+1}}{2\Delta\phi}\,\lambda_{i+1}^{n+1} - \frac{D_{k_i}^{n+1}}{2\Delta\phi}\,\Theta_{i-1}^{n+1} + \frac{C_{k_i}^{n+1}}{\Delta x}\,\Theta_{i}^{n+1} \\ &+ \frac{D_{k_i}^{n+1}}{2\Delta\phi}\,\Theta_{i+1}^{n+1} = - E_{k_i}^{n+1} + \frac{A_{k_i}^{n+1}}{\Delta x}\,\lambda_{i}^{n} + \frac{C_{k_i}^{n+1}}{\Delta x}\,\Theta_{i}^{n} \end{aligned} \tag{40}$$

where $k = 1, 2$.

This equation leads to a matrix equation of the form

$$H\,\overline{F} = \overline{C} \tag{41}$$

where $\overline{F}$ is the solution vector.

The size of H, $\overline{F}$, $\overline{C}$ are functions of the number of grid points, M, in the ϕ-direction. If the number of grid points in the ϕ-direction is M then H is a $2M$ by $2M$ matrix while $\overline{F}$ and $\overline{C}$ are $2M$ column vectors. The vector of unknowns, $\overline{F}$, is written as follows

$$\overline{F} = \begin{Bmatrix} \lambda_1^{n+1} \\ \lambda_2^{n+1} \\ \cdot \\ \cdot \\ \cdot \\ \lambda_{M-1}^{n+1} \\ \lambda_M^{n+1} \\ \Theta_1^{n+1} \\ \Theta_2^{n+1} \\ \cdot \\ \cdot \\ \cdot \\ \Theta_{M-1}^{n+1} \\ \Theta_M^{n+1} \end{Bmatrix}$$

Due to the conical nature of this grid $\lambda_{M+1}^{n+1} = \lambda_1^{n+1}$ and $\Theta_{M+1}^{n+1} = \Theta_1^{n+1}$.

The expressions for the elements of H, h(i,j), and $\overline{C}$, c(i), can be written more conveniently by defining the following

$$IP = \begin{cases} \frac{i}{2} & \text{if} \quad i \text{ is even} \\ \frac{i+1}{2} & \text{if} \quad i \text{ is odd} \end{cases}$$

$$IC = \begin{cases} 2 & \text{if} \quad i \text{ is even} \\ 1 & \text{if} \quad i \text{ is odd} \end{cases}$$

and

$$IM = IP - 1 \ .$$

Now c(i) may be written as follows

$$c(i) = E^{n+1}_{IC_{IP}} + \frac{A^{n+1}_{IC_{IP}}}{\Delta x} \lambda^n_{IP} + \frac{C^{n+1}_{IC_{IP}}}{\Delta x} \Theta^n_{IP}$$

$$\text{for any } i \quad .$$

To describe h(i,j), there are three different cases depending on the value of i. If $i \leq 2$ then h(i,j) may be written as follows

$$h(i,j) = \begin{cases} \dfrac{A^{n+1}_{IC_{IP}}}{\Delta x} & j = 1 \\[2ex] \dfrac{B^{n+1}_{IC_{IP}}}{2\Delta\phi} & j = 2 \\[2ex] -\dfrac{B^{n+1}_{IC_{IP}}}{2\Delta\phi} & j = M \\[2ex] \dfrac{C^{n+1}_{IC_{IP}}}{\Delta x} & j = M + 1 \\[2ex] \dfrac{D^{n+1}_{IC_{IP}}}{2\Delta\phi} & j = M + 2 \\[2ex] -\dfrac{D^{n+1}_{IC_{IP}}}{2\Delta\phi} & j = 2M \\[2ex] 0 & \text{for all other } j\text{'s} \ . \end{cases}$$

If $2 < i < 2M - 1$ then $h(i,j)$ may be written as follows

$$h(i,j) = \begin{cases} -\dfrac{B^{n+1}_{IC_{IP}}}{2\Delta\phi} & j = IM \\ \dfrac{A^{n+1}_{IC_{IP}}}{\Delta x} & j = IM + 1 \\ \dfrac{B^{n+1}_{IC_{IP}}}{2\Delta\phi} & j = IM + 2 \\ -\dfrac{D^{n+1}_{IC_{IP}}}{2\Delta\phi} & j = M + IM \\ \dfrac{C^{n+1}_{IC_{IP}}}{\Delta x} & j = M + IM + 1 \\ \dfrac{D^{n+1}_{IC_{IP}}}{2\Delta\phi} & j = M + IM + 2 \\ 0 & \text{for all other } j\text{'s} \end{cases}$$

and lastly, if $i \geq 2M - 1$ then $h(i,j)$ becomes

$$h(i,j) = \begin{cases} \dfrac{B_{IC_{IP}}^{n+1}}{2\Delta\phi} & j = 1 \\ -\dfrac{B_{IC_{IP}}^{n+1}}{2\Delta\phi} & j = M - 1 \\ \dfrac{A_{IC_{IP}}^{n+1}}{\Delta x} & j = M \\ \dfrac{D_{IC_{IP}}^{n+1}}{2\Delta\phi} & j = M + 1 \\ -\dfrac{D_{IC_{IP}}^{n+1}}{2\Delta\phi} & j = 2M - 1 \\ \dfrac{C_{IC_{IP}}^{n+1}}{\Delta x} & j = 2M \\ 0 & \text{for all other } j\text{'s} \end{cases} .$$

Since the coefficients

$$\left(A_{k_i}^{n+1},\ B_{k_i}^{n+1},\ C_{k_i}^{n+1},\ D_{k_i}^{n+1},\ E_{k_i}^{n+1},\ k = 1,\ 2\right)$$

are functions of the unknowns, a first guess of the unknowns at the (n + 1) line is used to evaluate the coefficients. To obtain this first guess a central difference explicit-finite method of the following form was used

$$\left.\frac{\partial\xi}{\partial x}\right|_i^n = \frac{\xi_i^{n+1} - \left(\xi_{i-1}^n + \xi_{i+1}^n\right)/2}{\Delta x}$$

and

$$\left.\frac{\partial\xi}{\partial\phi}\right|_i^n = \frac{\xi_{i+1}^n - \xi_{i-1}^n}{2\Delta\phi} .$$

Then equation (41) is solved to get a better approximation of the unknowns. Next the coefficients are reevaluated and the process is repeated until the unknowns from one step are sufficiently close to the unknowns of the next step. Note that the explicit forms above were used for the first guess only. The solution technique is an implicit one.

This implicit finite-difference method was found to be stable even at reasonably large step sizes. When applied to a cone at zero angle of attack, with or without spin, it was found that only one or two iterations were needed at each step to reach convergence. On a cone at angle of attack, this method usually required about three iterations to converge but could vary from one to ten iterations. The criterion used for convergence was a relative error of 0.1 percent.

5. RESULTS

The computer program was compiled in sections on an IBM 370/175 computer. The compiler used was the Fortran-H level compiler. The execution time varied depending on the case being tested.

5.1 Special Case of a Cone with Zero Spin at Zero Angle of Attack

First it was decided to test the theory against known solutions for zero spin at zero angle of attack of which there are many[10]. Two methods of solutions were used. One was with the full set of governing equations. The other method was to reduce the two governing partial differential equations to one ordinary differential equation since conditions of zero spin and zero angle of attack give $\frac{\partial \xi}{\partial \phi} = 0$ for any parameter ξ.

Notice that in the case of zero spin, $\omega = 0$, at zero angle of attack, $\alpha = 0$, the following results may be determined

$$\beta = 0 ,$$

$$\sigma = 0 ,$$

$$\varepsilon = 0 ,$$

$$a = 0 ,$$

and

$$\Theta = 0 .$$

Now apply these conditions to equation (33) to obtain the reduced equation

$$(G_1 - 3\,\alpha_s\, G_2)\,\frac{d\lambda}{dx} + \frac{\lambda}{U_e}\,\frac{dU_e}{dx}\left(\lambda^2\,\delta^+ - G_1\right) - \frac{\lambda^4}{R_L}\,G_2\,\frac{d^2\left(\frac{1}{U_e}\right)}{dx^2} + \frac{\lambda}{x}\,G_9$$

$$= U_e\,R_L \tag{42}$$

Since equation (42) is a first order ordinary differential equation, λ was found by using the fourth order Runge-Kutta integration technique[12]. The other method employed both partial differential equations (equations (33) and (34)), and the implicit finite-difference method mentioned previously. This check was performed as part of the debugging process for the computer program.

Solutions were calculated for a 10° semi-vertex cone in a flow of 60.96 m/s (200 ft/sec). The length of the cone was 1.2192 m (4 ft). Results of these solutions are shown in Figures 14 and 15. Figure 14 shows the boundary layer thickness for different skin friction jumps. Figure 15 shows the associated skin friction. These results were found to be satisfactory. The Runge-Kutta method took less than 10 seconds of execution time while the implicit finite-difference method took approximately 1 minute.

5.2 Special Case of a Cone at an Angle of Attack with Zero Spin

The next step is to introduce angle of attack to the cone. From the results of the zero spin at zero angle of attack case it was decided that a skin friction jump of $\frac{c_{f_{turb}}}{c_{f_{lam}}} = 1.96$ would be used because the solution using this value of the skin friction jump seemed to fit classical data best. For this case an angle of attack of 5° was used. The free stream velocity was again 60.96 m/s (200 ft/sec) on a cone of length 1.2192 m (4 ft) with a semi-vertex angle of 10°.

In order to show the instability caused by an explicit finite-difference scheme, Figures 16 through 18 show results when an explicit finite-

difference scheme was used to obtain solutions to the governing equations. In these figures, the solution started oscillating even though the step sizes satisfied the stability criterion. In Figure 16 a step size of $\Delta x = 0.005$, where x is non-dimensional length with respect to the length of cone, was used along with 40 grid points around the cone to produce a $\Delta\phi$ of 0.15708 radians or 9 degrees. Notice that the oscillation first occurred at an x-location of about 0.38. In Figure 17 everything was kept the same except the step size Δx which was changed to 0.01. This time the oscillation occurred at an x-location of about 0.6 which is better than in Figure 16. Finally in Figure 18 both step sizes were decreased to $\Delta x = 0.01$ and $\Delta\phi = 0.07854$ radians or 4.5 degrees, 80 grid points around the cone. The flow remained stable here until $x = 0.76$. Although the solution for this last case is nearly stable, the grid size, M, in the ϕ-direction being 80 points required a large amount of storage, 400k bytes, and a fairly large amount of execution time, over thirty minutes, on the IBM 370 computer to arrive at this solution.

At this point it was decided to change to the implicit finite-difference scheme mentioned earlier. The step sizes were held the same as in the explicit scheme to facilitate the comparison of these two methods. Results of the solutions using this implicit scheme are shown in Figures 19 through 21. Figure 19 shows the solution on a 10° cone at $\alpha = 5°$ without spin. The free stream velocity is again 60.96 m/s (200 ft/sec). The step size in the x-direction was 0.01 while the number of grid points in the ϕ-direction was 40, i.e., $\Delta\phi = 0.15708$ radians. Figure 19 shows the boundary layer thickness versus x at several different ϕ-locations. Figure 20 is similar to Figure 19 except Δx is changed to 0.02. In comparing Figures 19

and 20, it can be seen that Δx has very little effect on the solution obtained.

Finally, in Figure 21, the number of grid points in the ϕ-direction was changed to 20 to give a $\Delta\phi$ of 18 degrees or 0.314159 radians. Solutions were obtained for Δx of both 0.01 and 0.02. Changes in Δx had very little effect on the solutions obtained. The solution in Figure 21 is very similar to the solutions shown in Figures 19 and 20. A small difference can be noticed at ϕ = 180 degrees and this difference is in the curvature of the boundary layer thickness curve.

It is seen that a better solution was obtained with the larger step size in the ϕ-direction because of the curvature of the boundary layer thickness curve at ϕ = 180 degrees. These results show that the implicit finite-difference method is stable and is fairly insensitive to step size changes. Using the implicit method these solutions took approximately 2 minutes of execution time on the IBM 360 computer.

5.3 Cone at Angle of Attack with Spin

The case of the cone at angle of attack with spin is still being studied. Preliminary investigations have been performed but a solution has been obtained with a circular transition line only. The case of spin at zero angle of attack was tried first and the results are shown in Figures 22 and 23. The free stream velocity was again 60.96 m/s (200 ft/sec) on a 1.2192 m (4 ft) 10° cone. The step size in the x-direction was 0.01 while there were 20 grid points in the ϕ-direction. In Figure 22 the spin velocity was 500 revolutions per minute or 52.3 radians per second while in Figure 23 the spin rate was 400 revolutions per minute or 41.89 radians

per second. It can be seen from these results that Θ is more sensitive to spin rate than is the boundary layer thickness, δ. In comparing Figures 14, 22, and 23, there is very little change in δ due to the spin rate while in Figures 22 and 23 there is significant change in Θ.

The solution for the case of angle of attack with spin is shown in Figures 24 and 25. This solution is for the cone conditions as in Figure 23 except $\alpha = 5°$. Figure 24 shows the variation of boundary-layer thickness while Figure 25 shows the associated Θ. Since this solution is the only solution obtained with spin and angle of attack, no trends could be established. However, these initial results seem very promising and this work is continuing. Execution time for these cases was an average of 2 1/2 minutes.

6. CONCLUSIONS

The following conclusions were drawn from the analysis described in this report:

1. The numerical method developed here gives results which follow the trend of available experimental data and other analytical results.
2. When using an explicit finite-difference method to solve the governing equations, small step sizes are required to obtain stable and accurate results; the effect of small step sizes on the computer program is large execution times and large storage requirements.
3. The implicit method described in this report overcomes the step size problem and still remains stable and accurate.
4. Analysis of the governing equations showed them to be hyperbolic, but the characteristics were practically parallel to the cone generators and thus the characteristics crossed at large distances downstream.
5. Because of the large characteristic distances, the step sizes used were based on truncation error rather than on the zone of influence.
6. Large spin rates (cone base surface velocity larger than free stream velocity) would cause the computer solution to break down, particularly in the case of angle of attack.
7. Computer solutions can be found in the cases where the spin velocity produces a cone base surface velocity on the same order or less than free stream velocity.
8. Typical execution times for the implicit finite-difference method was less than three minutes on an IBM 370/175 computer.

7. RECOMMENDATIONS

It is recommended that future work be conducted to include the following:

1. the skewed transition line,
2. and a better method for the calculation of y_o^+, such as the method of Spalding and Chi[11].

APPENDIX A

Derivation of Governing Equations

First rearrange the continuity equation (eq. (7)) to get the following form

$$\frac{\partial v}{\partial y} = -\left[\frac{\partial u}{\partial s} + \frac{u}{s} + \frac{1}{s \sin\psi}\frac{\partial w}{\partial \phi}\right]$$

and then non-dimensionalize to obtain

$$\frac{1}{\nu_w}\frac{\partial v}{\partial y^+} = -\frac{\partial u^+}{\partial s} - \frac{1}{s}\left(1 + \frac{s}{u^*}\frac{\partial u^*}{\partial s}\right)u^+$$

$$- \frac{1}{s \sin\psi}\left(\frac{\partial w^+}{\partial \phi} + \frac{1}{u^*}\frac{\partial u^*}{\partial \phi}w^+\right) . \qquad \text{(A-1)}$$

Now use the velocity coorelations (eqs. (25) and (31)) and the definition of α_s (eq. (19)) to evaluate the above. First expend the derivative of the u velocity profile

$$\frac{\partial u^+}{\partial s} = \frac{\partial y^+}{\partial s}\frac{\partial u^+}{\partial y^+} + \frac{\partial \alpha_s}{\partial s}\frac{\partial u^+}{\partial \alpha_s} . \qquad \text{(A-2)}$$

Since $y^+ = \frac{u^* y}{\nu_w}$ then

$$\frac{\partial y^+}{\partial s} = \frac{y}{\nu_w}\frac{\partial u^*}{\partial s}$$

$$= \frac{y}{\nu_w}\frac{\partial}{\partial s}\left(\frac{u^*}{u_e}u_e\right)$$

$$= y^+\left(\frac{1}{u_e}\frac{\partial u_e}{\partial s} - \frac{1}{u_e^+}\frac{\partial u_e^+}{\partial s}\right) .$$

Define A_3 to be

$$A_3 = \frac{1}{u^*}\frac{\partial u^*}{\partial s} = \frac{1}{u_e}\frac{\partial u_e}{\partial s} - \frac{1}{u_e^+}\frac{\partial u_e^+}{\partial s} \tag{A-3}$$

such that

$$\frac{\partial y^+}{\partial s} = A_3\, y^+ \quad . \tag{A-4}$$

The derivative of eq. (19) with respect to S gives

$$\frac{\partial \alpha_s}{\partial s} = \frac{\nu_w}{(u^*)^3}\Bigg[-u_e\frac{\partial^2 u_e}{\partial s^2} - \left(\frac{\partial u_e}{\partial s}\right)^2 + \frac{w_e}{s}\left(\frac{\partial w_e}{\partial s}\right.$$

$$\left. - \frac{1}{\sin\psi}\frac{\partial^2 u_e}{\partial s\partial\phi}\right) + \left(\frac{1}{s}\frac{\partial w_e}{\partial s} - \frac{1}{s^2}w_e\right)\left(w_e\right.$$

$$\left. - \frac{1}{\sin\psi}\frac{\partial u_e}{\partial\phi}\right) - \omega^2\sin^2\psi\Bigg]$$

$$- \frac{\nu_w}{(u^*)^3}\frac{3}{u^*}\frac{\partial u^*}{\partial s}\Bigg[-\frac{w_e}{s}\left(w_e - \frac{1}{\sin\psi}\frac{\partial u_e}{\partial\phi}\right)$$

$$- \omega^2 s\sin^2\psi\Bigg] \tag{A-5}$$

Define the following

$$B_3 = \frac{1}{u^*}\frac{\partial u^*}{\partial\phi} = \frac{1}{u_e}\frac{\partial u_e}{\partial\phi} - \frac{1}{u_e^+}\frac{\partial u_e^+}{\partial\phi} \tag{A-6}$$

and $$A_4 = -\frac{1}{u_e^3}\Bigg[\frac{w_e}{s\sin\psi}\frac{\partial^2 u_e}{\partial s\partial\phi} + \left(1 + \frac{3s}{u_e}\frac{\partial u_e}{\partial s}\right)\left(w_e\right.$$

$$\left. - \frac{1}{\sin\psi}\frac{\partial u_e}{\partial\phi}\right)\frac{w_e}{s^2} + \omega^2\sin^2\psi\left(1 - \frac{3s}{u_e}\frac{\partial u_e}{\partial s}\right)$$

$$- \frac{w_e}{s}\frac{\partial w_e}{\partial s}\left(2 - \frac{1}{w_e\sin\psi}\frac{\partial u_e}{\partial\phi}\right)\Bigg] \tag{A-7}$$

and substitute into equation (A-5) to obtain

$$\frac{\partial \alpha_s}{\partial s} = \frac{3\alpha_s}{u_e^+}\frac{\partial u_e^+}{\partial s} + \nu_w\,(u_e^+)^3\left[\frac{\partial^2\left(\frac{1}{u_e}\right)}{\partial s^2} + A_4\right] \quad . \tag{A-8}$$

Substitute equations (A-4) and (A-8) into equation (A-2) to get

$$\frac{\partial u^+}{\partial s} = A_3\,y^+\,\frac{\partial u^+}{\partial y^+} + \left\{\frac{3\alpha_s}{u_e^+}\frac{\partial u_e^+}{\partial s} + \nu_w\,(u_e^+)^3 \left[\frac{\partial^2\left(\frac{1}{u_e}\right)}{\partial s^2} + A_4\right]\right\}\frac{\partial u^+}{\partial \alpha_s} \quad . \tag{A-9}$$

Next look at the w velocity component. Using the chain rule of differential calculus, the following derivative is obtained

$$\frac{\partial w^+}{\partial \phi} = \frac{\partial y^+}{\partial \phi}\frac{\partial w^+}{\partial y^+} + \frac{\partial \alpha_s}{\partial \phi}\frac{\partial w^+}{\partial \alpha_s} + \frac{\partial \beta}{\partial \phi}\frac{\partial w^+}{\partial \beta} + \frac{\partial \sigma}{\partial \phi}\frac{\partial w^+}{\partial \sigma} + \frac{\partial \Theta}{\partial \phi}\frac{\partial w^+}{\partial \Theta} + \frac{\partial \delta^+}{\partial \phi}\frac{\partial w^+}{\partial \delta^+} + \frac{\partial u_e^+}{\partial \phi}\frac{\partial w^+}{\partial u_e^+} \quad . \tag{A-10}$$

From the definition of y^+ it follows that

$$\frac{\partial y^+}{\partial \phi} = \frac{y}{\nu_w}\frac{\partial u^*}{\partial \phi} = y^+\,\frac{1}{u^*}\frac{\partial u^*}{\partial \phi} = B_3\,y^+ \quad . \tag{A-11}$$

The derivative of eq. (19) with respect to ϕ gives

$$\frac{\partial \alpha_s}{\partial \phi} = \frac{\nu_w}{u_3^*}\left[\left(-\,u_e\,\frac{\partial^2 u_e}{\partial_s \partial \phi} - \frac{\partial u_e}{\partial \phi}\frac{\partial u_e}{\partial s}\right) + \frac{w_e}{s}\left(\frac{\partial w_e}{\partial \phi} - \frac{1}{\sin\psi}\frac{\partial^2 u_e}{\partial \phi^2}\right) + \frac{1}{s}\frac{\partial w_e}{\partial \phi}\left(w_e - \frac{1}{\sin\psi}\frac{\partial u_e}{\partial \phi}\right)\right]$$

$$- \frac{\nu_w}{(u^*)^3} \frac{3}{u^*} \frac{\partial u^*}{\partial \phi} \left[- u_e \frac{\partial u_e}{\partial s} + \frac{w_e}{s} \left(w_e - \frac{1}{\sin \psi} \frac{\partial u_e}{\partial \phi} \right) \right.$$

$$\left. - \omega^2 s \sin^2 \psi \right] \quad . \tag{A-12}$$

Now define the following

$$B_4 = \frac{1}{u_e^3} \left[\frac{w_e}{s} \frac{\partial w_e}{\partial \phi} + \frac{1}{s} \frac{\partial w_e}{\partial \phi} \left(w_e - \frac{1}{\sin \psi} \frac{\partial u_e}{\partial \phi} \right) \right.$$

$$- \frac{w_e}{s \sin \psi} \frac{\partial^2 u_e}{\partial \phi^2} - \frac{3}{u_e} \frac{\partial u_e}{\partial \phi} \frac{w_e}{s} \left(w_e - \frac{1}{\sin \psi} \frac{\partial u_e}{\partial \phi} \right)$$

$$\left. + \frac{3 \omega^2 s \sin^2 \psi}{u_e} \frac{\partial u_e}{\partial \phi} \right] \tag{A-13}$$

and substitute into equation (A-12) to obtain

$$\frac{\partial \alpha_s}{\partial \phi} = \frac{3\alpha_s}{u_e^+} \frac{\partial u_e^+}{\partial \phi} + \nu_w (u_e^+)^3 \left[\frac{\partial^2 \left(\frac{1}{u_e} \right)}{\partial s \partial \phi} + B_4 \right] . \tag{A-14}$$

Rearrange equation (26) to get

$$\kappa u_e^+ = 2(q_{\delta^+} - 1) + \ln(q_{\delta^+} - 1) - \ln(q_{\delta^+} + 1)$$

$$+ \ln(4) - \ln(\alpha_s y_o^+)$$

then

$$\kappa \frac{\partial u_e^+}{\partial \phi} = 2 \frac{\partial q_{\delta^+}}{\partial \phi} + \frac{1}{q_{\delta^+}-1} \frac{\partial q_{\delta^+}}{\partial \phi} - \frac{1}{q_{\delta^+}-1} \frac{\partial q_{\delta^+}}{\partial \phi} - \frac{1}{\alpha_s y_o^+} y_o^+ \frac{\partial \alpha_s}{\partial \phi}$$

and rearrange the above to obtain

$$\kappa \frac{\partial u_e^+}{\partial \phi} = \frac{\alpha_s q_{\delta^+}}{q_{\delta^+}^2 - 1} \frac{\partial \delta^+}{\partial \phi} + \left[\frac{q_{\delta^+} \delta^+}{q_{\delta^+}^2 - 1} - \frac{1}{\alpha_s} \right] \frac{\partial \alpha_s}{\partial \phi}$$

and finally solve for $\frac{\partial \delta^+}{\partial \phi}$ to obtain the following

$$\frac{\partial \delta^+}{\partial \phi} = \frac{\kappa \delta^+}{q_{\delta^+}} \frac{\partial u_e^+}{\partial \phi} - \frac{\delta^+ (q_{\delta^+} - 1)}{\alpha_s q_{\delta^+}} \frac{\partial \alpha_s}{\partial \phi} \quad . \tag{A-15}$$

Next substitute equations (A-11), (A-14), and (A-15) into equation (A-10) to get the following form

$$\frac{\partial w^+}{\partial \phi} = B_3\, y^+ \frac{\partial w^+}{\partial y^+} + \left\{ \frac{3\alpha_s}{u_e^+} \frac{\partial u_e^+}{\partial \phi} + \nu_w\, (u_e^+)^3 \left[\frac{\partial^2 \left(\frac{1}{u_e}\right)}{\partial s \partial \phi} + B_4 \right] \right\} \left[\frac{\partial w^+}{\partial \alpha_s} - \frac{\delta^+ (q_{\delta^+} - 1)}{\alpha_s\, q_{\delta^+}} \frac{\partial w^+}{\partial \delta^+} \right]$$

$$+ \frac{\partial u_e^+}{\partial \phi} \left(\frac{\kappa \delta^+}{q_{\delta^+}} \frac{\partial w^+}{\partial \delta^+} \right) + \frac{\partial \Theta}{\partial \phi} \frac{\partial w^+}{\partial \Theta} + \frac{\partial \beta}{\partial \phi} \frac{\partial w^+}{\partial \beta} + \frac{\partial \sigma}{\partial \phi} \frac{\partial w^+}{\partial \sigma} + \sigma \frac{\partial u_e^+}{\partial \phi} . \qquad \text{(A-16)}$$

Substitute equations (A-9) and (A-16) into equation (A-1) to get the continuity equation in the following form

$$- \frac{1}{\nu_w} \frac{\partial v}{\partial y^+} = A_3\, y^+ \frac{\partial u^+}{\partial y^+} + \left\{ \frac{3\alpha_s}{u_e^+} \frac{\partial u_e^+}{\partial s} + \nu_w\, (u_e^+)^3 \left[\frac{\partial^2 \left(\frac{1}{u_e}\right)}{\partial s^2} + A_4 \right] \right\} \frac{\partial u^+}{\partial \alpha_s} + \frac{1}{s} (1 + A_3\, s)\, u^+$$

$$+ \frac{B_3}{s \sin \psi} w^+ + \frac{1}{s \sin \psi} \left\{ B_3\, y^+ \frac{\partial w^+}{\partial y^+} + \left\{ \frac{3\alpha_s}{u_e^+} \frac{\partial u_e^+}{\partial \phi} + \nu_w\, (u_e^+)^3 \left[\frac{\partial^2 \left(\frac{1}{u_e}\right)}{\partial s \partial \phi} + B_4 \right] \right\} \left[\frac{\partial w^+}{\partial \alpha_s} - \frac{\delta^+ (q_{\delta^+} - 1)}{\alpha_s\, q_{\delta^+}} \frac{\partial w^+}{\partial \delta^+} \right] \right.$$

$$+ \frac{\partial u_e^+}{\partial \phi} \left(\frac{\kappa \delta^+}{q_{\delta^+}} \frac{\partial w^+}{\partial \delta^+} \right) + \frac{\partial \Theta}{\partial \phi} \frac{\partial w^+}{\partial \Theta} + \frac{\partial \beta}{\partial \phi} \frac{\partial w^+}{\partial \beta}$$

$$\left. + \frac{\partial \sigma}{\partial \phi} \frac{\partial w^+}{\partial \sigma} + \sigma \frac{\partial u_e^+}{\partial \phi} \right\} . \qquad \text{(A-17)}$$

Define the following integral terms

$$I_0 = \int_0^{y^+} u^+ \, dy^+ \quad \text{(A-18)}$$

$$I_1 = \int_0^{y^+} y^+ \frac{\partial u^+}{\partial y^+} dy^+ = (y^+ u^+)_0^{y^+} - \int_0^{y^+} u^+ \, dy^+$$

$$= y^+ u^+ - I_0 \quad \text{(A-19)}$$

$$I_2 = \int_0^{y^+} \frac{\partial u^+}{\partial \alpha_s} dy^+ \quad \text{(A-20)}$$

$$J_0 = \int_0^{y^+} w^+ \, dy^+ \quad \text{(A-21)}$$

$$J_1 = \int_0^{y^+} y^+ \frac{\partial w^+}{\partial y^+} dy^+ = (y^+ w^+)_0^{y^+} - \int_0^{y^+} w^+ \, dy^+$$

$$= y^+ w^+ - J_0 \quad \text{(A-22)}$$

$$J_2 = \int_0^{y^+} \frac{\partial w^+}{\partial \alpha_s} dy^+ \quad \text{(A-23)}$$

$$J_3 = \int_0^{y^+} \frac{\partial w^+}{\partial \delta^+} dy^+ \quad \text{(A-24)}$$

$$J_4 = \int_0^{y^+} \frac{\partial w^+}{\partial \Theta} dy^+ \quad \text{(A-25)}$$

$$J_5 = \int_0^{y^+} \frac{\partial w^+}{\partial \beta} dy^+ \quad \text{(A-26)}$$

$$\text{and}\quad J_6 = \int_0^{y^+} \frac{\partial w^+}{\partial \sigma}\, dy^+ \ . \tag{A-27}$$

Multiply equation (A-17) by dy^+, and integrate, and substitute equations (A-18) through (A-27) to obtain the final form of the continuity equation

$$- \frac{v}{\nu_w} = A_3\, y^+ u^+ + \frac{1}{s} I_0 + \left\{ \frac{3\alpha_s}{u_e^+} \frac{\partial u_e^+}{\partial s} + \nu_w \,(u_e^+)^3 \left[\frac{\partial^2 \left(\frac{1}{u_e}\right)}{\partial s^2} + A_4 \right] \right\} I_2$$

$$+ \frac{1}{s \sin \psi} \left\{ B_3\, y^+ w^+ + \left[\frac{3\alpha_s}{u_e^+} \frac{\partial u_e^+}{\partial \phi} + \nu_w \,(u_e^+)^3 \left(\frac{\partial^2 \left(\frac{1}{u_e}\right)}{\partial_s \partial \phi} + B_4 \right) \right] \right.$$

$$\left[J_2 - \frac{\delta + (q_{\delta^+} - 1)}{\alpha_s q_{\delta^+}} J_3 \right] + \frac{\partial u_e^+}{\partial \phi} \frac{\kappa \delta^+}{q_{\delta^+}} J_3$$

$$\left. + \frac{\partial \Theta}{\partial \phi} J_4 + \frac{\partial \beta}{\partial \phi} J_5 + \frac{\partial \sigma}{\partial \phi} J_6 + \frac{\partial u_e^+}{\partial \phi} \sigma y^+ \right\} . \tag{A-28}$$

Now consider the conservation of momentum in the s-direction (equation (8)) which can be rearranged and substitutions made from equations (A-3) and (A-6) to obtain

$$u^+ \frac{\partial u^+}{\partial s} + \frac{w^+}{s \sin \psi} \frac{\partial u^+}{\partial \phi} + \frac{v}{\nu_w} \frac{\partial u^+}{\partial y^+} + A_3\, (u^+)^2$$

$$+ B_3 \frac{u^+ w^+}{s \sin \psi} - \frac{(w^+)^2}{s}$$

$$= u_e^+ \frac{\partial u_e^+}{\partial s} + \frac{w_e^+}{s \sin \psi} \frac{\partial u_e^+}{\partial \phi} + A_3\, (u_e^+)^2 + B_3 \frac{u_e^+ w_e^+}{s \sin \psi}$$

$$- \frac{(w_e^+)^2}{s} + \frac{u^*}{\nu_w} \frac{\partial}{\partial y^+} (\tau_s / \tau_{s_w}) \ . \tag{A-29}$$

Use the u-velocity profile and equation (A-14) to obtain

$$\frac{\partial u^+}{\partial \phi} = \frac{\partial y^+}{\partial \phi}\frac{\partial u^+}{\partial y^+} + \frac{\partial \alpha_s}{\partial \phi}\frac{\partial u^+}{\partial \alpha_s}$$

$$= B_3\, y^+ \frac{\partial u^+}{\partial y^+} + \left\{ \frac{3\alpha_s}{u_e^+}\frac{\partial u_e^+}{\partial \phi} + \nu_w \left(u_e^+\right)^3 \left[\frac{\partial^2\left(\frac{1}{u_e}\right)}{\partial s\partial \phi} + B_4\right]\right\}\frac{\partial u^+}{\partial \alpha_s} \quad . \tag{A-30}$$

Substitute equations (A-9), (A-28) and (A-30) into equation (A-29) and rearrange to obtain the following

$$\left\{ \frac{3\alpha_s}{u_e^+}\frac{\partial u_e^+}{\partial s} + \nu_w \left(u_e^+\right)^3 \left[\frac{\partial^2\left(\frac{1}{u_e}\right)}{\partial s^2} + A_4\right]\right\}\left[u^+ \frac{\partial u^+}{\partial \alpha_s} - \frac{\partial u^+}{\partial y^+} I_2\right] + \frac{1}{s \sin \psi}\left\{ \frac{3\alpha_s}{u_e^+}\frac{\partial u_e^+}{\partial \phi} + \nu_w \left(u_e^+\right)^3 \left[\frac{\partial^2\left(\frac{1}{u_e}\right)}{\partial s\partial \phi} + B_4\right]\right\}$$

$$\left[w^+ \frac{\partial u^+}{\partial \alpha_s} - \frac{\partial u^+}{\partial y^+} J_2 + \frac{\delta^+(q_{\delta^+}-1)}{\alpha_s q_{\delta^+}}\frac{\partial u^+}{\partial y^+} J_3\right]$$

$$- \frac{1}{s}\left[\left(w^+\right)^2 + \frac{\partial u^+}{\partial y^+} I_0\right] + \left[\frac{\partial u_e^+}{\partial \phi}\frac{\kappa\delta^+}{q_{\delta^+}}\frac{\partial u^+}{\partial y^+} J_3\right.$$

$$\left. - \frac{\partial \Theta}{\partial \phi}\frac{\partial u^+}{\partial y^+} J_4 - \frac{\partial \beta}{\partial \phi}\frac{\partial u^+}{\partial y^+} J_5 - \frac{\partial \sigma}{\partial \phi}\frac{\partial u^+}{\partial y^+} J_6 - \frac{\partial u_e^+}{\partial \phi}\sigma\, y^+ \frac{\partial u^+}{\partial y^+}\right]\frac{1}{s \sin \psi}$$

$$+ A_3 \left(u^+\right)^2 + B_3 \frac{u^+ w^+}{s \sin \psi}$$

$$= u_e^+ \frac{\partial u_e^+}{\partial s} + \frac{w_e^+}{s \sin\psi} \frac{\partial u_e^+}{\partial \phi} + A_3 (u_e^+)^2 + B_3 \frac{u_e^+ w_e^+}{s \sin\psi} - \frac{(w_e^+)^2}{s}$$

$$+ \frac{u_e}{\nu_w} \frac{1}{u_e^+} \frac{\partial}{\partial y^+} (\tau_s/\tau_{s_w}) \; . \tag{A-31}$$

Integrate equation (A-31) from $y^+ = 0$ to $y^+ = \delta^+$ and define

$$G_1 = \int_0^{\delta^+} (u^+)^2 \, dy^+ \tag{A-32}$$

$$G_2 = \int_0^{\delta^+} \left[u^+ \frac{\partial u^+}{\partial \alpha_s} - \frac{\partial u^+}{\partial y^+} I_2 \right] dy^+ \tag{A-33}$$

$$G_3 = \int_0^{\delta^+} u^+ \, w^+ \, dy^+ \tag{A-34}$$

$$G_4 = \int_0^{\delta^+} \left[w^+ \frac{\partial u^+}{\partial \alpha_s} - \frac{\partial u^+}{\partial y^+} J_2 + \frac{\delta^+ (q_{\delta^+} - 1)}{\alpha_s q_{\delta^+}} \frac{\partial u^+}{\partial y^+} J_3 \right] dy^+ \tag{A-35}$$

$$G_5 = \int_0^{\delta^+} \frac{\kappa \delta^+}{q_{\delta^+}} \frac{\partial u^+}{\partial y^+} J_3 \, dy^+ \tag{A-36}$$

$$G_6 = \int_0^{\delta^+} \frac{\partial u^+}{\partial y^+} J_4 \, dy^+ \tag{A-37}$$

$$G_7 = \int_0^{\delta^+} \frac{\partial u^+}{\partial y^+} J_5 \, dy^+ \tag{A-38}$$

$$G_8 = \int_0^{\delta^+} \frac{\partial u^+}{\partial y^+} J_6 \, dy^+ \tag{A-39}$$

$$G_9 = \int_0^{\delta^+} \left[(w^+)^2 + \frac{\partial u^+}{\partial y^+} I_0 \right] dy^+ \tag{A-40}$$

and

$$\int_0^{\delta^+} y^+ \frac{\partial u^+}{\partial y^+} dy^+ = (y^+ u^+)_0^{\delta^+} - \int_0^{\delta^+} u^+ dy^+ = \delta^+ u_e^+ - I_0^{(e)} \tag{A-41}$$

to obtain the following

$$\begin{aligned}
&\frac{\partial u_e^+}{\partial s} (G_1 - 3\alpha_s G_2) + \frac{\partial u_e^+}{\partial \phi} \frac{1}{s \sin \psi} \Bigg[G_3 - 3\alpha_s G_4 + u_e^+ G_5 \\
&\quad + \sigma u_e^+ (\delta^+ u_e^+ - I_0^{(e)}) \Bigg] + \frac{\partial \Theta}{\partial \phi} \frac{u_e^+ G_6}{s \sin \psi} \\
&\quad + \frac{u_e^+}{u_e} \frac{\partial u_e}{\partial s} \left[(u_e^+)^2 \delta^+ - G_1 \right] + \frac{u_e^+}{u_e} \frac{\partial u_e}{\partial \phi} \frac{1}{s \sin \psi} (u_e^+ w_e^+ \delta^+ \\
&\quad - G_3 - \beta G_7 - \sigma G_8) + \frac{u_e^+}{s \sin \psi} \frac{\beta}{w_e} \frac{\partial w_e}{\partial \phi} G_7 \\
&\quad - \nu_w (u_e^+)^4 \left\{ G_2 \left[\frac{\partial^2 \left(\frac{1}{u_e} \right)}{\partial s^2} + A_4 \right] + \frac{G_4}{s \sin \psi} \left[\frac{\partial^2 \left(\frac{1}{u_e} \right)}{\partial s \partial \phi} \right. \right. \\
&\quad \left. \left. + B_4 \right] \right\} + \frac{u_e^+}{s} (G_9 - \delta^+ (w_e^+)^2) = \frac{u_e}{\nu_w} \; .
\end{aligned} \tag{A-42}$$

Define the following non-dimensional parameters

$$x = s/L \tag{A-43}$$

$$\lambda = u_e^+ \tag{A-44}$$

$$U_e = \frac{u_e}{u_\infty} \tag{A-45}$$

$$W_e = \frac{w_e}{u_\infty} \tag{A-46}$$

$$R_L = \frac{u_\infty L}{\nu_w} \tag{A-47}$$

$$\varepsilon^2 = \frac{\omega^2 L^2}{u_\infty^2} \tag{A-48}$$

and then define

$$a = -\frac{1}{{U_e}^3}\left[\frac{W_e}{x \sin\psi}\frac{\partial^2 U_e}{\partial x \partial\phi} + \left(1 + \frac{3x}{U_e}\frac{\partial U_e}{\partial x}\right)\left(W_e - \frac{1}{\sin\psi}\frac{\partial U_e}{\partial\phi}\right)\frac{W_e}{x^2} + \varepsilon^2 \sin^2\psi\left(1 - \frac{3x}{U_e}\frac{\partial U_e}{\partial x}\right) - \frac{W_e}{x}\frac{\partial W_e}{\partial x}\left(2 - \frac{1}{W_e \sin\psi}\frac{\partial U_e}{\partial\phi}\right)\right] \tag{A-49}$$

and

$$b = \frac{1}{{U_e}^3}\left(\frac{W_e}{x}\right)\left[\frac{\partial W_e}{\partial\phi} - \frac{1}{\sin\psi}\frac{\partial^2 U_e}{\partial\phi^2} + \left(W_e - \frac{1}{\sin\psi}\frac{\partial U_e}{\partial\phi}\right)\left(\frac{1}{W_e}\frac{\partial W_e}{\partial\phi} - \frac{3}{U_e}\frac{\partial U_e}{\partial\phi}\right)\right] + \frac{3\,\varepsilon^2 \times \sin^2\psi}{U_e^4}\frac{\partial U_e}{\partial\phi} \tag{A-50}$$

Define the following non-dimensional parameters

$$x = s/L \tag{A-43}$$

$$\lambda = u_e^+ \tag{A-44}$$

$$U_e = \frac{u_e}{u_\infty} \tag{A-45}$$

$$W_e = \frac{w_e}{u_\infty} \tag{A-46}$$

$$R_L = \frac{u_\infty L}{\nu_w} \tag{A-47}$$

$$\varepsilon^2 = \frac{\omega^2 L^2}{u_\infty^2} \tag{A-48}$$

and then define

$$a = -\frac{1}{U_e^{\,3}}\left[\frac{W_e}{x\sin\psi}\frac{\partial^2 U_e}{\partial x \partial \phi} + \left(1 + \frac{3x}{U_e}\frac{\partial U_e}{\partial x}\right)\left(W_e - \frac{1}{\sin\psi}\frac{\partial U_e}{\partial \phi}\right)\frac{W_e}{x^2} + \varepsilon^2 \sin^2\psi \left(1 - \frac{3x}{U_e}\frac{\partial U_e}{\partial x}\right) - \frac{W_e}{x}\frac{\partial W_e}{\partial x}\left(2 - \frac{1}{W_e \sin\psi}\frac{\partial U_e}{\partial \phi}\right)\right] \tag{A-49}$$

and

$$b = \frac{1}{U_e^{\,3}}\left(\frac{W_e}{x}\right)\left[\frac{\partial W_e}{\partial \phi} - \frac{1}{\sin\psi}\frac{\partial^2 U_e}{\partial \phi^2} + \left(W_e - \frac{1}{\sin\psi}\frac{\partial U_e}{\partial \phi}\right)\left(\frac{1}{W_e}\frac{\partial W_e}{\partial \phi} - \frac{3}{U_e}\frac{\partial U_e}{\partial \phi}\right)\right] + \frac{3\,\varepsilon^2 \times \sin^2\psi}{U_e^4}\frac{\partial U_e}{\partial \phi} \tag{A-50}$$

so that

$$A_4 = \frac{1}{u_\infty L^2} \; a \tag{A-51}$$

$$B_4 = \frac{1}{u_\infty L} \; b \tag{A-52}$$

and then substitute these parameters into equation (A-42) to obtain the following

$$(G_1 - 3\alpha_s\, G_2)\, \frac{\partial \lambda}{\partial x} + \frac{1}{x \sin \psi} \Bigg[G_3 - 3\alpha_s\, G_4 + \lambda\, G_5$$

$$+ \sigma\, \lambda\, (\delta^+ \lambda - I_0^{(e)}) \Bigg] \frac{\partial \lambda}{\partial \phi} + \frac{\lambda G_6}{x \sin \psi}\, \frac{\partial \Theta}{\partial \phi} + \frac{\lambda}{U_e}\, \frac{\partial U_e}{\partial x}$$

$$(\lambda^2\, \delta^+ - G_1) + \frac{\lambda}{U_e}\, \frac{\partial U_e}{\partial \phi}\, \frac{1}{x \sin \psi}\, (\beta\, \lambda^2\, \delta^+ - G_3 - \beta\, G_7$$

$$- \sigma\, G_8) + \frac{\lambda}{x \sin \psi}\, \frac{\beta}{W_e}\, \frac{\partial W_e}{\partial \phi}\, G_7$$

$$- \frac{\lambda^4}{R_L} \left\{ G_2 \left[\frac{\partial^2 \left(\frac{1}{U_e} \right)}{\partial x^2} + a \right] + \frac{G_4}{x \sin \psi} \left[\frac{\partial^2 \left(\frac{1}{U_e} \right)}{\partial x \partial \phi} + b \right] \right\}$$

$$+ \frac{\lambda}{x}\, (G_9 - \beta^2\, \delta^+\, \lambda^2) = U_e\, R_L \tag{A-53}$$

which is the final form of the s-momentum equation and the first governing equation.

Next consider the conservation of momentum in the ϕ-direction (equation (9)) which can be rearranged and substitutions made from equations (A-3) and (A-6) to obtain

$$u^+ \left[\frac{\partial w^+}{\partial s} + w^+ A_3 \right] + \frac{w^+}{s \sin \psi} \left[\frac{\partial w^+}{\partial \phi} + w^+ B_3 \right]$$

$$+ \frac{v}{\nu_w} \frac{\partial w^+}{\partial y^+} + \frac{u^+ w^+}{s} = u_e^+ \left[\frac{\partial w_e^+}{\partial s} + w_e^+ A_3 \right]$$

$$+ \frac{w_e^+}{s \sin \psi} \left[\frac{\partial w_e^+}{\partial \phi} + w_e^+ B_3 \right] + \frac{u_e^+ w_e^+}{s} + \frac{\Theta}{u_e^+} \frac{u_e}{\nu_w} \frac{\partial}{\partial y^+} (\tau_\phi / \tau_{\phi_w}) . \tag{A-54}$$

First expand the derivative of the w-velocity correlation by the chain rule as follows

$$\frac{\partial w^+}{\partial s} = \frac{\partial y^+}{\partial s} \frac{\partial w^+}{\partial y^+} + \frac{\partial \alpha_s}{\partial s} \frac{\partial w^+}{\partial \alpha_s} + \frac{\partial \beta}{\partial s} \frac{\partial w^+}{\partial \beta}$$

$$+ \frac{\partial \sigma}{\partial s} \frac{\partial w^+}{\partial \sigma} + \frac{\partial \Theta}{\partial s} \frac{\partial w^+}{\partial \Theta} + \frac{\partial \delta^+}{\partial s} \frac{\partial w^+}{\partial \delta^+} + \frac{\partial u_e^+}{\partial s} \frac{\partial w^+}{\partial u_e^+} . \tag{A-55}$$

Rearrange equation (26) to get

$$\kappa \, u_e^+ = 2(q_{\delta^+} - 1) + \ln(q_{\delta^+} - 1) - \ln(q_{\delta^+} + 1)$$

$$+ \ln(4) - \ln(\alpha_s \, y_o^+)$$

then take the derivative with respect to s and rearrange to obtain

$$\kappa \frac{u_e^+}{\partial s} = \frac{q_{\delta^+}}{\delta^+} \frac{\partial \delta^+}{\partial s} + \left(\frac{q_{\delta^+}}{\alpha_s} - \frac{1}{\alpha_s} \right) \frac{\partial \alpha_s}{\partial s} .$$

Then solve for $\frac{\partial \delta^+}{\partial s}$ to obtain the following

$$\frac{\partial \delta^+}{\partial s} = \frac{\kappa \delta^+}{q_{\delta^+}} \frac{\partial u_e^+}{\partial s} - \frac{\delta^+ (q_{\delta^+} - 1)}{\alpha_s q_{\delta^+}} \frac{\partial \alpha_s}{\partial s} \quad . \tag{A-56}$$

Substitute equations (A-4), (A-8), and (A-56) into equation (A-55) to get

$$\frac{\partial w^+}{\partial s} = A_3 \, y^+ \frac{\partial w^+}{\partial y^+} + \left\{ \frac{3\alpha_s}{u_e^+} \frac{\partial u_e^+}{\partial s} + \nu_w \, (u_e^+)^3 \left[\frac{\partial^2 \left(\frac{1}{u_e}\right)}{\partial s^2} \right.\right.$$

$$\left.\left. + A_4 \right] \right\} \left[\frac{\partial w^+}{\partial \alpha_s} - \frac{\delta^+ (q_{\delta^+} - 1)}{\alpha_s q_{\delta^+}} \frac{\partial w^+}{\partial \delta^+} \right]$$

$$+ \frac{\partial u_e^+}{\partial s} \left(\frac{\kappa \delta^+}{q_{\delta^+}} \frac{\partial w^+}{\partial \delta^+} + \sigma \right) + \frac{\partial \Theta}{\partial s} \frac{\partial w^+}{\partial \Theta}$$

$$+ \beta \left(\frac{1}{w_e} \frac{\partial w_e}{\partial s} - \frac{1}{u_e} \frac{\partial u_e}{\partial s} \right) \frac{\partial w^+}{\partial \beta} + \frac{\sigma}{s} \left(1 - \frac{s}{u_e} \frac{\partial u_e}{\partial s} \right) \frac{\partial w^+}{\partial \sigma} \quad . \tag{A-57}$$

Now substitute equations (A-16), (A-28), and (A-57) in to equation (A-54) and rearrange to obtain the following

$$\left\{ \frac{3\alpha_s}{u_e^+} \frac{\partial u_e^+}{\partial s} + \nu_w \, (u_e^+)^3 \left[\frac{\partial^2 \left(\frac{1}{u_e}\right)}{\partial s} + A_4 \right] \right\} \left[u^+ \frac{\partial w^+}{\partial \alpha_s} \right.$$

$$\left. - \frac{\partial w^+}{\partial y^+} I_2 - \frac{\delta^+ (q_{\delta^+} - 1)}{\alpha_s q_{\delta^+}} u^+ \frac{\partial w^+}{\partial \delta^+} \right]$$

$$+ \frac{1}{s \sin \psi} \left\{ \frac{3\alpha_s}{u_e^+} \frac{\partial u_e^+}{\partial \phi} + \nu_w \, (u_e^+)^3 \left[\frac{\partial^2 \left(\frac{1}{u_e}\right)}{\partial s \partial \phi} + B_4 \right] \right\} \left[w^+ \frac{\partial w^+}{\partial \alpha_s} \right.$$

$$- J_2 \frac{\partial w^+}{\partial y^+} - \frac{\delta^+ (q_{\delta^+} - 1)}{\alpha_s q_{\delta^+}} \Bigl(w^+ \frac{\partial w^+}{\partial \delta^+} - J_3 \frac{\partial w^+}{\partial y^+} \Bigr) \Bigr] + \sigma \frac{\partial u_e^+}{\partial s} u^+$$

$$+ \frac{\partial u_e^+}{\partial s} \frac{\kappa \delta^+}{q_{\delta^+}} u^+ \frac{\partial w^+}{\partial \delta^+} + \frac{\partial \Theta}{\partial s} \Bigl(u^+ \frac{\partial w^+}{\partial \Theta} \Bigr) + \beta \Bigl(\frac{1}{w_e} \frac{\partial w_e}{\partial s}$$

$$- \frac{1}{u_e} \frac{\partial u_e}{\partial s} \Bigr) \Bigl(u^+ \frac{\partial w^+}{\partial \beta} \Bigr) + \frac{\sigma}{s} \Bigl(1 - \frac{s}{u_e} \frac{\partial u_e}{\partial s} \Bigr) u^+ \frac{\partial w^+}{\partial \sigma}$$

$$+ \frac{1}{s \sin \psi} \frac{\partial u_e^+}{\partial \phi} \frac{\kappa \delta^+}{q_{\delta^+}} \Bigl[w^+ \frac{\partial w^+}{\partial \delta^+} - J_3 \frac{\partial w^+}{\partial y^+} \Bigr] + \frac{1}{s \sin \psi} \frac{\partial \Theta}{\partial \phi}$$

$$\Bigl[w^+ \frac{\partial w^+}{\partial \Theta} - J_4 \frac{\partial w^+}{\partial y^+} \Bigr] + \frac{1}{s \sin \psi} \beta \Bigl(\frac{1}{w_e} \frac{\partial w_e}{\partial \phi}$$

$$- \frac{1}{u_e} \frac{\partial u_e}{\partial \phi} \Bigr) \Bigl[w^+ \frac{\partial w^+}{\partial \beta} - J_5 \frac{\partial w^+}{\partial y^+} \Bigr] - \frac{1}{s \sin \psi} \frac{\sigma}{u_e} \frac{\partial u_e}{\partial \phi} \Bigl[w^+ \frac{\partial w^+}{\partial \sigma}$$

$$- J_6 \frac{\partial w^+}{\partial y^+} \Bigr] + \frac{\sigma}{s \sin \psi} \frac{\partial u_e^+}{\partial \phi} \Bigl[w^+ - y^+ \frac{\partial w^+}{\partial y^+} \Bigr]$$

$$+ \frac{1}{s} \Bigl(u^+ w^+ - \frac{\partial w^+}{\partial y^+} I_0 \Bigr) + A_3 u^+ w^+ + \frac{B_3}{s \sin \psi} (w^+)^2$$

$$= \frac{w_e^+ u_e^+}{w_e} \frac{\partial w_e}{\partial s} + \frac{1}{s \sin \psi} \frac{(w_e^+)^2}{w_e} \frac{\partial w_e}{\partial \phi} + \frac{u_e^+ w_e^+}{s}$$

$$+ \frac{u_e}{u_e^+} \frac{\Theta}{\nu_w} \frac{\partial}{\partial y^+} (\tau_\phi / \tau_{\phi_w}) \; . \qquad \text{(A-58)}$$

Integrate the above equation from $y^+ = 0$ to $y^+ = \delta^+$ and define

$$G_{10} = \int_0^{\delta^+} \left[u^+ \frac{\partial w^+}{\partial \alpha_s} - \frac{\partial w^+}{\partial y^+} I_2 - \frac{\delta^+(q_{\delta^+} - 1)}{\alpha_s q_{\delta^+}} u^+ \frac{\partial w^+}{\partial \delta^+} \right] dy^+ \quad \text{(A-59)}$$

$$G_{11} = \frac{\kappa \delta^+}{q_{\delta^+}} \int_0^{\delta^+} u^+ \frac{\partial w^+}{\partial \delta^+} dy^+ \quad \text{(A-60)}$$

$$G_{12} = \int_0^{\delta^+} (w^+)^2 \, dy^+ \quad \text{(A-61)}$$

$$G_{13} = \int_0^{\delta^+} \left[w^+ \frac{\partial w^+}{\partial \alpha_s} - \frac{\partial w^+}{\partial y^+} J_2 - \frac{\delta^+(q_{\delta^+} - 1)}{\alpha_s q_{\delta^+}} \left(w^+ \frac{\partial w^+}{\partial \delta^+} - \frac{\partial w^+}{\partial y^+} J_3 \right) \right] dy^+ \quad \text{(A-62)}$$

$$G_{14} = \frac{\kappa \delta^+}{q_{\delta^+}} \int_0^{\delta^+} \left[w^+ \frac{\partial w^+}{\partial \delta^+} - \frac{\partial w^+}{\partial y^+} J_3 \right] dy^+ \quad \text{(A-63)}$$

$$G_{15} = \int_0^{\delta^+} u^+ \frac{\partial w^+}{\partial \Theta} dy^+ \quad \text{(A-64)}$$

$$G_{16} = \int_0^{\delta^+} \left[w^+ \frac{\partial w^+}{\partial \Theta} - \frac{\partial w^+}{\partial y^+} J_4 \right] dy^+ \quad \text{(A-65)}$$

$$G_{17} = \int_0^{\delta^+} \left[u^+ w^+ - \frac{\partial w^+}{\partial y^+} I_0 \right] dy^+ \quad \text{(A-66)}$$

$$G_{18} = \int_0^{\delta^+} u^+ \frac{\partial w^+}{\partial \sigma} \, dy^+ \tag{A-67}$$

$$G_{19} = \int_0^{\delta^+} \left[w^+ \frac{\partial w^+}{\partial \sigma} - \frac{\partial w^+}{\partial y^+} J_6 \right] dy^+ \tag{A-68}$$

$$G_{20} = \int_0^{\delta^+} u^+ \frac{\partial w^+}{\partial \beta} \, dy^+ \tag{A-69}$$

$$G_{21} = \int_0^{\delta^+} \left[w^+ \frac{\partial w^+}{\partial \beta} - \frac{\partial w^+}{\partial y^+} J_5 \right] dy^+ \tag{A-70}$$

to obtain the following

$$\frac{\partial u_e^+}{\partial s} \left[G_3 - 3\alpha_s \, G_{10} - u_e^+ G_{11} - \sigma \, u_e^+ I_0^{(e)} \right]$$

$$+ \frac{1}{s \sin \psi} \frac{\partial u_e^+}{\partial \phi} \left[G_{12} - 3\alpha_s \, G_{13} - u_e^+ G_{14} - \sigma \, u_e^+ (2 \, J_0^{(e)} - \delta^+ w_e^+) \right]$$

$$- \frac{\partial \Theta}{\partial s} (u_e^+ G_{15}) - \frac{1}{s \sin \psi} \frac{\partial \Theta}{\partial \phi} (u_e^+ G_{16}) + \frac{u_e^+}{u_e} \frac{\partial u_e}{\partial s} (\sigma \, G_{18}$$

$$+ \beta \, G_{20} - G_3) + \frac{u_e^+}{u_e} \frac{\partial u_e}{\partial \phi} \frac{1}{s \sin \psi} (\sigma \, G_{19} + \beta \, G_{21} - G_{12})$$

$$+ \frac{u_e^+}{w_e} \frac{\partial w_e}{\partial s} \beta \left[(u_e^+)^2 \delta^+ - G_{20} \right] + \frac{1}{s \sin \psi} \frac{u_e^+}{w_e} \frac{\partial w_e}{\partial \phi} \beta$$

$$\left[\beta\,(u_e^+)^2\,\delta^+ - G_{21}\right] - \nu_w\,(u_e^+)^4\left\{\left[\frac{\partial^2\left(\frac{1}{u_e}\right)}{\partial s^2} + A_4\right] G_{10} + \frac{1}{s\sin\psi}\left[\frac{\partial^2\left(\frac{1}{u_e}\right)}{\partial s\partial\phi} + B_4\right] G_{13}\right\} + \frac{u_e^+}{s}\left[\beta\,(u_e^+)^2\,\delta^+ - G_{17} - \sigma\, G_{18}\right] = \frac{u_e}{\nu_w}\,\Theta\ . \qquad \text{(A-71)}$$

Multiply the above equation by L and then substitute equations (A-43) through (A-50) to get

$$\frac{\partial\lambda}{\partial x}\left(G_3 - 3\alpha_s\, G_{10} - \lambda\, G_{11} - \sigma\,\lambda\, I_0^{(e)}\right) + \frac{1}{x\sin\psi}\frac{\partial\lambda}{\partial\phi}\left[G_{12} - 3\alpha_s\, G_{13} - \lambda\, G_{14} - \sigma\,\lambda\,(2\,J_0^{(e)} - \delta^+\,\beta\,\lambda)\right]$$

$$- G_{15}\,\lambda\,\frac{\partial\Theta}{\partial x} - \frac{1}{x\sin\psi}\,G_{16}\,\lambda\,\frac{\partial\Theta}{\partial\phi} + \frac{\lambda}{U_e}\frac{\partial U_e}{\partial\phi}\frac{1}{x\sin\psi}\left(\sigma\, G_{19} + \beta\, G_{21} - G_{12}\right) + \frac{\lambda}{W_e}\frac{\partial W_e}{\partial x}\,\beta\,[\lambda^2\,\delta^+ - G_{20}]$$

$$+ \frac{1}{x\sin\psi}\frac{\lambda}{W_e}\frac{\partial W_e}{\partial\phi}\,\beta\,[\beta\,\lambda^2\,\delta^+ - G_{21}] + \frac{\lambda}{U_e}\frac{\partial U_e}{\partial x}\left(\sigma\, G_{18} + \beta\, G_{20} - G_3\right)$$

$$- \frac{\lambda^4}{R_L}\left\{\left[\frac{\partial^2\left(\frac{1}{U_e}\right)}{\partial x^2} + a\right] G_{10} + \frac{1}{x\sin\psi}\left[\frac{\partial^2\left(\frac{1}{U_e}\right)}{\partial x\partial\phi} + b\right] G_{13}\right\}$$

$$+ \frac{\lambda}{x}\left(\beta\,\lambda^2\,\delta^+ - G_{17} - \sigma\, G_{18}\right) = U_e\, R_L\,\Theta \qquad \text{(A-72)}$$

which is the final form of the ϕ-momemtum equation and the second governing equation.

APPENDIX B

Evaluation of Basic Integrals

The evaluation of the basic integrals needed to evaluate the coefficients in the governing equations is given below.

$$\int_0^{y^+} y^+ \, dy^+ = \frac{1}{2} y^{+2} \qquad \int_0^{\delta^+} y^+ \, dy^+ = \frac{1}{2} \delta^{+2}$$

$$\int_0^{y^+} y^{+2} \, dy^+ = \frac{1}{3} y^{+3} \qquad \int_0^{\delta^+} y^{+2} \, dy^+ = \frac{1}{3} \delta^{+3}$$

$$\int_0^{y^+} q \, dy^+ = \frac{2}{3\,\alpha_s}(q^3 - 1) \qquad \int_0^{\delta^+} q \, dy^+ = \frac{2}{3\,\alpha_s}(q^3_{\delta^+} - 1)$$

$$\int_0^{\delta^+} q^2 \, dy^+ = \frac{1}{2\,\alpha_s}(q^4_{\delta^+} - 1) \qquad \int_0^{\delta^+} q^3 \, dy^+ = \frac{2}{5\,\alpha_s}(q^5_{\delta^+} - 1)$$

$$\int_0^{\delta^+} q^4 \, dy^+ = \frac{1}{3\,\alpha_s}(q^6_{\delta^+} - 1) \qquad \int_0^{\delta^+} q^5 \, dy^+ = \frac{2}{7\,\alpha_s}(q^7_{\delta^+} - 1)$$

$$\int_0^{\delta^+} q^6 \, dy^+ = \frac{1}{4\,\alpha_s}(q^8_{\delta^+} - 1) \qquad \int_0^{\delta^+} q^7 \, dy^+ = \frac{2}{9\,\alpha_s}(q^9_{\delta^+} - 1)$$

$$\int_0^{\delta^+} q^8 \, dy^+ = \frac{1}{5\,\alpha_s}(q^{10}_{\delta^+} - 1) \qquad \int_0^{\delta^+} q^9 \, dy^+ = \frac{2}{11\,\alpha_s}(q^{11}_{\delta^+} - 1)$$

$$\int_0^{\delta^+} q^{10}\, dy^+ = \frac{1}{6\,\alpha_s}\left(q_{\delta^+}^{12} - 1\right)$$

$$\int_0^{y^+} u^+\, dy^+ = y^+ u^+ - \frac{2}{3}\frac{1}{\kappa\alpha_s}\left(q^3 - 1\right)$$

$$\int_0^{\delta^+} u^+\, dy = \delta^+ u_e^+ - \frac{2}{3}\frac{1}{\kappa\alpha_s}\left(q_{\delta^+}^3 - 1\right)$$

$$\int_0^{\delta^+} u^{+2}\, dy^+ = \delta^+ u_e^{+2} - \frac{4}{3\,\kappa\alpha_s}\left[u_e^+\left(q_{\delta^+}^3 - 1\right) - \frac{1}{\kappa}\int_0^{\delta^+}\frac{q}{y^+}\left(q^3 - 1\right) dy^+\right]$$

$$\int_0^{\delta^+}\frac{q}{y^+}\left(q^3 - 1\right) dy^+ = 2\left(q_{\delta^+}^4 + q_{\delta^+}^3 + q_{\delta^+}^2\right)\ln\left(q_{\delta^+} + 1\right)$$

$$- 6\ln 2 - 4\int_1^{q_{\delta^+}} q\ln(q + 1)\, dq$$

$$- 6\int_1^{q_{\delta^+}} q^2\ln(q + 1)\, dq$$

$$- 8\int_1^{q_{\delta^+}} q^3\ln(q + 1)\, dq$$

$$\int_0^{\delta^+}\frac{q}{y^+}\left(2q^3 - 3q^2 + 1\right) dy^+ = 2\left(2q_{\delta^+}^4 - q_{\delta^+}^3 - q_{\delta^+}^2\right)\ln\left(q_{\delta^+} + 1\right)$$

$$+ 4 \int_1^{q_{\delta^+}} q \ln (q + 1)\, dq$$

$$+ 6 \int_1^{q_{\delta^+}} q^2 \ln (q + 1)\, dq$$

$$- 16 \int_1^{q_{\delta^+}} q^3 \ln (q + 1)\, dq$$

$$\int_0^{\delta^+} \frac{q}{y^+} (3q^5 - 5q^3 + 2)\, dy^+ = 2(3q_{\delta^+}^6 + 3q_{\delta^+}^5 - 2q_{\delta^+}^4 - 2q_{\delta^+}^3$$

$$- 2q_{\delta^+}^2) \ln (q + 1)$$

$$+ 8 \int_1^{q_{\delta^+}} q \ln (q + 1)\, dq + 12 \int_1^{q_{\delta^+}} q^2 \ln (q + 1)\, dq$$

$$+ 16 \int_1^{q_{\delta^+}} q^3 \ln (q + 1)\, dq - 30 \int_1^{q_{\delta^+}} q^4 \ln (q + 1)\, dq$$

$$- 36 \int_1^{q_{\delta^+}} q^5 \ln (q + 1)\, dq$$

$$\int_0^{\delta^+} \frac{q}{y^+} (12q^5 - 15q^4 - 20q^3 + 30q^2 - 7)\, dy^+ =$$

$$2(12q_{\delta^+}^6 - 3q_{\delta^+}^5 - 23q_{\delta^+}^4 + 7q_{\delta^+}^3 + 7q_{\delta^+}^2) \ln (q + 1)$$

$$- 28 \int_1^{q_{\delta^+}} q \ln (q + 1)\, dq - 42 \int_1^{q_{\delta^+}} q^2 \ln (q + 1)\, dq$$

$$+ 184 \int_1^{q_{\delta^+}} q^3 \ln (q + 1)\, dq + 30 \int_1^{q_{\delta^+}} q^4 \ln (q + 1)\, dq$$

$$- 144 \int_1^{q_{\delta^+}} q^5 \ln (q + 1)\, dq$$

$$\int_0^{\delta^+} \frac{q}{y^+} (q^6 - 3q^4 + 3q^2 - 1)\, dy^+ = 2(q_{\delta^+}^7 + q_{\delta^+}^6 - 2q_{\delta^+}^5$$

$$- 2q_{\delta^+}^4 + q_{\delta^+}^3 + q_{\delta^+}^2) \ln (q_{\delta^+} + 1)$$

$$- 4 \int_1^{q_{\delta^+}} q \ln (q + 1)\, dq - 6 \int_1^{q_{\delta^+}} q^2 \ln (q + 1)\, dq$$

$$+ 16 \int_1^{q_{\delta^+}} q^3 \ln (q + 1)\, dq + 20 \int_1^{q_{\delta^+}} q^4 \ln (q + 1)\, dq$$

$$- 12 \int_1^{q_{\delta^+}} q^5 \ln (q + 1) - 14 \int_1^{q_{\delta^+}} q^6 \ln (q + 1)\, dq$$

$$\int_0^{\delta^+} \frac{q}{y^+} (15q^7 - 42q^5 + 35q^3 - 8)\, dy^+ = 2(15q_{\delta^+}^8 + 15q_{\delta^+}^7$$

$$- 27q_{\delta^+}^6 - 27q_{\delta^+}^5 + 8q_{\delta^+}^4 + 8q_{\delta^+}^3$$

$$+ 8q_{\delta^+}^2) \ln (q_{\delta^+} + 1) - 32 \int_1^{q_{\delta^+}} q \ln (q + 1)\, dq$$

$$- 48 \int_{1}^{q_{\delta^+}} q^2 \ln (q + 1) \, dq - 64 \int_{1}^{q_{\delta^+}} q^3 \ln (q + 1)$$

$$+ 270 \int_{1}^{q_{\delta^+}} q^4 \ln (q + 1) \, dq + 324 \int_{1}^{q_{\delta^+}} q^5 \ln (q + 1) \, dq$$

$$- 210 \int_{1}^{q_{\delta^+}} q^6 \ln (q + 1) \, dq - 240 \int_{1}^{q_{\delta^+}} q^7 \ln (q + 1) \, dq$$

$$\int_{0}^{y^+} y^+ \, q \, dy^+ = \frac{2}{15 \, \alpha_s^2} \, (3q^5 - 5q^3 + 2)$$

$$\int_{0}^{\delta^+} y^+ \, q \, dy^+ = \frac{2}{15 \, \alpha_s^2} \, (3q_{\delta^+}^{5} - 5_{\delta^+}^{3} + 2)$$

$$\int_{0}^{\delta^+} y^+ \, q^3 \, dy^+ = \frac{2}{35 \, \alpha_s^2} \, (5q_{\delta^+}^{7} - 7q_{\delta^+}^{5} + 2)$$

$$\int_{0}^{\delta^+} y^+ \, q^4 \, dy^+ = \frac{1}{12 \, \alpha_s^2} \, (3q_{\delta^+}^{8} - 4q_{\delta^+}^{6} + 1)$$

$$\int_{0}^{\delta^+} y^+ \, q^5 \, dy^+ = \frac{2}{63 \, \alpha_s^2} \, (7q_{\delta^+}^{9} - 9q_{\delta^+}^{7} + 2)$$

$$\int_{0}^{\delta^+} y^+ \, q^6 \, dy^+ = \frac{1}{20 \, \alpha_s^2} \, (4q_{\delta^+}^{10} - 5q_{\delta^+}^{8} + 1)$$

$$\int_{0}^{\delta^+} y^+ \, q^7 \, dy^+ = \frac{2}{99 \, \alpha_s^2} \, (9q_{\delta^+}^{11} - 11q_{\delta^+}^{9} + 2)$$

$$\int_0^{\delta^+} y^+ q^8\, dy = \frac{1}{30\,\alpha_s^2}\left(5q_{\delta^+}^{12} - 6q_{\delta^+}^{10} + 1\right)$$

$$\int_0^{y^+} y^{+2} q\, dy^+ = \frac{2}{105\,\alpha_s^3}\left(15q^7 - 42q^5 + 35q^3 - 8\right)$$

$$\int_0^{\delta^+} y^{+2} q\, dy^+ = \frac{2}{105\,\alpha_s^3}\left(15q_{\delta^+}^{7} - 42q_{\delta^+}^{5} + 35q_{\delta^+}^{3} - 8\right)$$

$$\int_0^{\delta^+} q\, u^+ dy^+ = \frac{2}{3\,\alpha_s}\left[u_e^+\left(q_{\delta^+}^{3} - 1\right) - \frac{1}{\kappa}\int_0^{\delta^+} \frac{q}{y^+}\left(q^3 - 1\right) dy^+\right]$$

$$\int_0^{\delta^+} y^+ u^+ dy^+ = \frac{1}{2}\delta^+ u_e^+ - \frac{1}{15\,\kappa\alpha_s^2}\left(3q_{\delta^+}^{5} - 5q_{\delta^+}^{3} + 2\right)$$

$$\int_0^{\delta^+} y^{+2} u^+ dy^+ = \frac{1}{3}\delta^{+3} u_e^+ - \frac{2}{315\,\kappa\alpha_s^3}\left(15q_{\delta^+}^{7} - 42q_{\delta^+}^{5} + 35q_{\delta^+}^{3} - 8\right)$$

$$\int_0^{\delta^+} y^{+3} u^+ dy^+ = \frac{1}{4}\delta^{+4} u_e^+ - \frac{1}{12\,\kappa\alpha_s^4}\left[\frac{2}{3}q_{\delta^+}^{9} - \frac{18}{7}q_{\delta^+}^{7} + \frac{18}{5}q_{\delta^+}^{5} - 2q_{\delta^+}^{3} + \frac{32}{105}\right]$$

$$\int_0^{\delta^+} y^{+4} u^+ dy^+ = \frac{1}{5}\delta^{+5} u_e^+ - \frac{1}{5\,\kappa\alpha_s^5}\left[\frac{2}{11}q_{\delta^+}^{11} - \frac{8}{9}q_{\delta^+}^{9} + \frac{12}{7}q_{\delta^+}^{7} - \frac{8}{5}q_{\delta^+}^{5} + \frac{2}{3}q_{\delta^+}^{3} - \frac{256}{3465}\right]$$

$$\int_0^{\delta^+} y^+ u^{+2} dy^+ = \frac{1}{2}\delta^{+2} u_e^{+2} - \frac{1}{\kappa}\int_0^{\delta^+} y^+ q\, u^+ dy^+$$

$$\int_0^{\delta+} y^{+2} u^{+2} dy^+ = \frac{1}{3} \delta^{+3} u_e^{+2} - \frac{2}{3\kappa} \int_0^{\delta^+} y^{+2} q u^+ dy^+$$

$$\int_0^{\delta^+} y^{+3} u^{+2} dy^+ = \frac{1}{4} \delta^{+4} u_e^{+2} - \frac{1}{2\kappa} \int_0^{\delta^+} y^{+3} q u^+ dy^+$$

$$\int_0^{\delta^+} y^{+4} u^{+2} dy^+ = \frac{1}{5} \delta^{+5} u_e^{+2} - \frac{2}{5\kappa} \int_0^{\delta^+} y^{+4} q u^+ dy^+$$

$$\int_0^{\delta^+} q^2 u^+ dy^+ = \frac{1}{2} (q_{\delta^+}^2 + 1) \delta^+ u_e^+ - \frac{1}{15 \kappa\alpha_s} (3q_{\delta^+}^5 + 5q_{\delta^+}^3 - 8)$$

$$\int_0^{\delta^+} q^3 u^+ dy^+ = q_{\delta^+}^3 \delta^+ u_e^+ - \frac{1}{3 \kappa\alpha_s} (q_{\delta^+}^6 - 1) - \frac{3 \alpha_s}{2} \int_0^{\delta^+} y^+ q u^+ dy^+$$

$$\int_0^{\delta^+} q^4 u^+ dy^+ = q_{\delta^+}^4 \delta^+ u_e^+ - \frac{2}{7 \kappa\alpha_s} (q_{\delta^+}^7 - 1) - 2 \alpha_s \int_0^{\delta^+} y^+ q^2 u^+ dy^+$$

$$\int_0^{\delta^+} q^5 u^+ dy^+ = q_{\delta^+}^5 \delta^+ u_e^+ - \frac{1}{4 \kappa\alpha_s} (q_{\delta^+}^8 - 1) - \frac{5 \alpha_s}{2} \int_0^{\delta^+} y^+ q^3 u^+ dy^+$$

$$\int_0^{\delta^+} q^6 u^+ dy^+ = q_{\delta^+}^6 \delta^+ u_e^+ - \frac{2}{9 \kappa\alpha_s} (q_{\delta^+}^9 - 1) - 3 \alpha_s \int_0^{\delta^+} y^+ q^4 u^+ dy^+$$

$$\int_0^{\delta^+} q^7 u^+ dy^+ = q_{\delta^+}^7 \delta^+ u_e^+ - \frac{1}{5 \kappa\alpha_s} (q_{\delta^+}^{10} - 1) - \frac{7 \alpha_s}{2} \int_0^{\delta^+} y^+ q^5 u^+ dy^+$$

$$\int_0^{y^+} y^{+3} q\, dy^+ = \frac{1}{315 \alpha_s^4} \left[70q^9 - 270q^7 + 378q^5 - 210q^3 + 32\right]$$

$$\int_0^{y^+} y^{+4} q\, dy^+ = \frac{1}{3465 \alpha_s^5} \left[630q^{11} - 3080q^9 + 5940q^7 - 5544q^5 + 2310q^3 - 256\right]$$

$$\int_0^{\delta^+} y^+ \, q \, u^+ \, dy^+ = \frac{2}{15 \, \alpha_s^2} \Bigg[u_e^+ \, (3q_{\delta^+}^{5} - 5q_{\delta^+}^{3} + 2)$$

$$+ \frac{1}{\kappa} \int_0^{\delta^+} \frac{q}{y^+} \, (3q^5 - 5q^3 + 2) \, dy^+ \Bigg]$$

$$\int_0^{\delta^+} y^{+2} \, q \, u^+ \, dy^+ = \frac{2}{105 \, \alpha_s^3} \Bigg[u_e^+ \, (15q_{\delta^+}^{7} - 42q_{\delta^+}^{5} + 35q_{\delta^+}^{3} - 8)$$

$$+ \frac{1}{\kappa} \int_0^{\delta^+} \frac{q}{y^+} \, (15q^7 - 42q^5 + 35q^3 - 8) \, dy^+ \Bigg]$$

$$\int_0^{\delta^+} y^{+3} \, q \, u^+ \, dy^+ = \frac{1}{315 \, \alpha_s^4} \Bigg[u_e^+ \, (70_{\delta^+}^{9} - 270q_{\delta^+}^{7} + 378_{\delta^+}^{5} - 210q_{\delta^+}^{3} + 32)$$

$$+ \frac{1}{\kappa} \int_0^{\delta^+} \frac{q}{y^+} \, (70q^9 - 270q^7 + 378q^5 - 210q^3 + 32) \, dy^+ \Bigg]$$

$$\int_0^{\delta^+} y^{+4} \, q \, u^+ \, dy^+ = \frac{1}{3465 \, \alpha_s^5} \Bigg[u_e^+ \, (630q_{\delta^+}^{11} - 3080q_{\delta^+}^{9} + 5940q_{\delta^+}^{7}$$

$$- 5544q_{\delta^+}^{5} + 2310q_{\delta^+}^{3} - 256) - \frac{1}{\kappa} \int_0^{\delta^+} \frac{q}{y^+} \, (630q^{11}$$

$$- 3080q^9 + 5940q^7 - 5544q^5 + 2310q^3 - 256) \, dy^+ \Bigg]$$

$$\int_0^{y^+} y^+ \, q^2 \, dy^+ = \frac{2}{12 \, \alpha_s^2} \, [2q^6 - 3q^4 + 1]$$

$$\int_0^{\delta^+} y^+ q^3 u^+ dy^+ = \frac{2}{35\,\alpha_s^2}\left[u_e^+(5q_{\delta^+}^7 - 7q_{\delta^+}^5 + 2)\right.$$

$$\left. + \frac{1}{\kappa}\int_0^{\delta^+} \frac{q}{y^+}(5q^7 - 7q^5 + 2)\, dy^+\right]$$

$$\int_0^{\delta^+} y^+ q^5 u^+ dy^+ = \frac{2}{63\,\alpha_s^2}\left[u_e^+ (7q_{\delta^+}^9 - 9q_{\delta^+}^7 + 2)\right.$$

$$\left. + \frac{1}{\kappa}\int_0^{\delta^+} \frac{q}{y^+}(7q^9 - 9q^7 + 2)\, dy^+\right]$$

$$\int_0^{\delta^+} y^+ q^7 u^+ dy^+ = \frac{2}{99\,\alpha_s^2}\left[u_e^+ (9q_{\delta^+}^{11} - 11q_{\delta^+}^9 + 2)\right.$$

$$\left. - \frac{1}{\kappa}\int_0^{\delta^+} \frac{q}{y^+}(9q^{11} - 11q^9 + 2)\, dy^+\right]$$

$$\int_0^{\delta^+} \frac{q}{y^+}(70q^9 - 270q^7 + 378q^5 - 210q^3 + 32)\, dy^+$$

$$= 2q_{\delta^+}^2 (70q_{\delta^+}^8 + 70q_{\delta^+}^7 - 200q_{\delta^+}^6 - 200q_{\delta^+}^5 + 178q_{\delta^+}^4$$

$$+ 178q_{\delta^+}^3 - 32q_{\delta^+}^2 - 32q_{\delta^+} - 32) \ln (q_{\delta^+} + 1)$$

$$+ 128\int_1^{q_{\delta^+}} q \ln (q + 1)\, dq + 192\int_1^{q_{\delta^+}} q^2 \ln (q + 1)\, dq$$

$$+ 256\int_1^{q_{\delta^+}} q^3 \ln (q + 1)\, dq - 1780\int_1^{q_{\delta^+}} q^4 \ln (q + 1)\, dq$$

$$- 2136 \int_{1}^{q_{\delta^+}} q^5 \ln (q + 1)\, dq + 2800 \int_{1}^{q_{\delta^+}} q^6 \ln (q + 1)\, dq$$

$$+ 3200 \int_{1}^{q_{\delta^+}} q^7 \ln (q + 1)\, dq - 1260 \int_{1}^{q_{\delta^+}} q^8 \ln (q + 1)\, dq$$

$$- 1400 \int_{1}^{q_{\delta^+}} q^9 \ln (q + 1)\, dq$$

$$\int_{0}^{\delta^+} \frac{q}{y^+} (630q^{11} - 3080q^9 + 5940q^7 - 5544q^5 + 2310q^3 - 256)\, dy^+$$

$$= 2q_{\delta^+}^2 (630q_{\delta^+}^{10} + 630q_{\delta^+}^9 - 2450q_{\delta^+}^8 - 2450q_{\delta^+}^7$$

$$+ 3490q_{\delta^+}^6 + 3490q_{\delta^+}^5 - 2054q_{\delta^+}^4 - 2054q_{\delta^+}^3$$

$$+ 256q_{\delta^+}^2 + 256q_{\delta^+} + 256) \ln (q_{\delta^+} + 1)$$

$$- 1024 \int_{1}^{q_{\delta^+}} q \ln (q + 1)\, dq - 1536 \int_{1}^{q_{\delta^+}} q^2 \ln (q + 1)\, dq$$

$$- 2048 \int_{1}^{q_{\delta^+}} q^3 \ln (q + 1)\, dq + 20540 \int_{1}^{q_{\delta^+}} q^4 \ln (q + 1)\, dq$$

$$- 24648 \int_{1}^{q_{\delta^+}} q^5 \ln (q + 1)\, dq - 48860 \int_{1}^{q_{\delta^+}} q^6 \ln (q + 1)\, dq$$

$$- 55840 \int_{1}^{q_{\delta^+}} q^7 \ln (q + 1)\, dq + 44100 \int_{1}^{q_{\delta^+}} q^8 \ln (q + 1)\, dq$$

$$+ 49000 \int_1^{q_{\delta^+}} q^9 \ln (q + 1)\, dq - 13860 \int_1^{q_{\delta^+}} q^{10} \ln (q + 1)\, dq$$

$$- 15120 \int_1^{q_{\delta^+}} q^{11} \ln (q + 1)\, dq$$

$$\int_0^{\delta^+} \frac{q}{y^+} (2q^6 - 3q^4 + 1)\, dy^+ = 2q_{\delta^+}^2 (2q_{\delta^+}^5 + 2q_{\delta^+}^4 - q_{\delta^+}^3 - q_{\delta^+}^2$$

$$- q_{\delta^+} -1) \ln (q_{\delta^+} - 1) + 4 \int_1^{q_{\delta^+}} q \ln (q + 1)\, dq$$

$$+ 6 \int_1^{q_{\delta^+}} q^2 \ln (q + 1)\, dq + 8 \int_1^{q_{\delta^+}} q^3 \ln (q + 1)\, dq$$

$$+ 10 \int_1^{q_{\delta^+}} q^4 \ln (q + 1)\, dq - 24 \int_1^{q_{\delta^+}} q^5 \ln (q + 1)\, dq$$

$$- 28 \int_1^{q_{\delta^+}} q^6 \ln (q + 1)\, dq$$

$$\int_0^{\delta^+} \frac{q}{y^+} (5q^7 - 7q^5 + 2)\, dy^+ = 2q_{\delta^+}^2 (5q_{\delta^+}^6 + 5q_{\delta^+}^5 - 2q_{\delta^+}^4 - 2q_{\delta^+}^3$$

$$- 2q_{\delta^+}^2 - 2q_{\delta^+} - 2) \ln (q_{\delta^+} + 1) + 8 \int_1^{q_{\delta^+}} q \ln (q + 1)\, dq$$

$$+ 12 \int_1^{q_{\delta^+}} q^2 \ln (q + 1)\, dq + 16 \int_1^{q_{\delta^+}} q^3 \ln (q + 1)\, dq$$

$$+ 20 \int_1^{q_{\delta^+}} q^4 \ln (q + 1)\, dq + 24 \int_1^{q_{\delta^+}} q^5 \ln (q + 1)\, dq$$

$$- 70 \int_1^{q_{\delta^+}} q^6 \ln (q + 1)\, dq - 80 \int_1^{q_{\delta^+}} q^7 \ln (q + 1)\, dq$$

$$\int_0^{\delta^+} \frac{q}{y^+} (3q^8 - 4q^6 + 1)\, dy^+ = 2q_{\delta^+}^2 (3q_{\delta^+}^7 + 3q_{\delta^+}^6 - q_{\delta^+}^5 - q_{\delta^+}^4 - q_{\delta^+}^3$$

$$- q_{\delta^+}^2 - q_{\delta^+} - 1) \ln (q_{\delta^+} + 1) + 4 \int_1^{q_{\delta^+}} q \ln (q + 1)\, dq$$

$$+ 6 \int_1^{q_{\delta^+}} q^2 \ln (q + 1)\, dq + 8 \int_1^{q_{\delta^+}} q^3 \ln (q + 1)\, dq$$

$$+ 10 \int_1^{q_{\delta^+}} q^4 \ln (q + 1)\, dq + 12 \int_1^{q_{\delta^+}} q^5 \ln (q + 1)\, dq$$

$$+ 14 \int_1^{q_{\delta^+}} q^6 \ln (q + 1)\, dq - 48 \int_1^{q_{\delta^+}} q^7 \ln (q + 1)\, dq$$

$$- 54 \int_1^{q_{\delta^+}} q^8 \ln (q + 1)\, dq$$

$$\int_0^{\delta^+} \frac{q}{y^+} (7q^9 - 9q^7 + 2)\, dy^+ = 2q_{\delta^+}^2 (7q_{\delta^+}^8 + 7q_{\delta^+}^7 - 2q_{\delta^+}^6$$

$$- 2q_{\delta^+}^5 - 2q_{\delta^+}^4 - 2q_{\delta^+}^3 - 2q_{\delta^+}^2 - 2q_{\delta^+} - 2) \ln (q + 1)$$

$$+ 8 \int_1^{q_{\delta^+}} q \ln (q + 1)\, dq + 12 \int_1^{q_{\delta^+}} q^2 \ln (q + 1)\, dq$$

$$+ 16 \int_1^{q_{\delta^+}} q^3 \ln (q + 1)\, dq + 20 \int_1^{q_{\delta^+}} q^4 \ln (q + 1)\, dq$$

$$+ 24 \int_1^{q_{\delta^+}} q^5 \ln (q + 1)\, dq + 28 \int_1^{q_{\delta^+}} q^6 \ln (q + 1)\, dq$$

$$+ 32 \int_1^{q_{\delta^+}} q^7 \ln (q + 1)\, dq - 126 \int_1^{q_{\delta^+}} q^8 \ln (q + 1)\, dq$$

$$- 140 \int_1^{q_{\delta^+}} q^9 \ln (q + 1)\, dq$$

$$\int_0^{\delta^+} \frac{q}{y^+} (4q^{10} - 5q^8 + 1)\, dy^+ = 2q_{\delta^+}^2 (4q_{\delta^+}^9 + 4q_{\delta^+}^8 - q_{\delta^+}^7 - q_{\delta^+}^6$$

$$- q_{\delta^+}^5 - q_{\delta^+}^4 - q_{\delta^+}^3 - q_{\delta^+}^2 - q_{\delta^+} - 1) \ln (q_{\delta^+} + 1)$$

$$+ 4 \int_1^{q_{\delta^+}} q \ln (q + 1)\, dq + 6 \int_1^{q_{\delta^+}} q^2 \ln (q + 1)\, dq$$

$$+ 8 \int_1^{q_{\delta^+}} q^3 \ln (q + 1)\, dq + 10 \int_1^{q_{\delta^+}} q^4 \ln (q + 1)\, dq$$

$$+ 12 \int_1^{q_{\delta^+}} q^5 \ln (q + 1)\, dq + 14 \int_1^{q_{\delta^+}} q^6 \ln (q + 1)\, dq$$

$$+ 16 \int_1^{q_{\delta^+}} q^7 \ln (q + 1)\, dq + 18 \int_1^{q_{\delta^+}} q^8 \ln (q + 1)\, dq$$

$$- 80 \int_1^{q_{\delta^+}} q^9 \ln (q + 1)\, dq - 88 \int_1^{q_{\delta^+}} q^{10} \ln (q + 1)\, dq$$

$$\int_0^{\delta^+} \frac{q}{y^+} (9q^{11} - 11q^9 + 2)\, dy^+ = 2q_{\delta^+}^{2} (9q_{\delta^+}^{10} + 9q_{\delta^+}^{9} - 2q_{\delta^+}^{8} - 2q_{\delta^+}^{7}$$

$$- 2q_{+}^{6} - 2q_{\delta^+}^{5} - 2q_{\delta^+}^{4} - 2q_{\delta^+}^{3} - 2q_{\delta^+}^{2} - 2q_{\delta^+} - 2) \ln (q_{\delta^+} + 1)$$

$$+ 8 \int_1^{q_{\delta^+}} q \ln (q + 1)\, dq + 12 \int_1^{q_{\delta^+}} q^2 \ln (q + 1)\, dq$$

$$+ 16 \int_1^{q_{\delta^+}} q^3 \ln (q + 1)\, dq + 20 \int_1^{q_{\delta^+}} q^4 \ln (q + 1)\, dq$$

$$+ 24 \int_1^{q_{\delta^+}} q^5 \ln (q + 1)\, dq + 28 \int_1^{q_{\delta^+}} q^6 \ln (q + 1)\, dq$$

$$+ 32 \int_1^{q_{\delta^+}} q^7 \ln (q + 1)\, dq + 36 \int_1^{q_{\delta^+}} q^8 \ln (q + 1)\, dq$$

$$+ 40 \int_1^{q_{\delta^+}} q^9 \ln (q + 1)\, dq - 198 \int_1^{q_{\delta^+}} q^{10} \ln (q + 1)\, dq$$

$$- 216 \int_1^{q_{\delta^+}} q^{11} \ln (q + 1)\, dq$$

$$\int_1^{q_{\delta^+}} q \ln (q + 1)\, dq = \frac{1}{2} \left[(q_{\delta^+}^{2} - 1) \ln (q_{\delta^+} + 1) - \frac{1}{2} (q_{\delta^+}^{2} - 2q_{\delta^+} + 1) \right]$$

$$\int_1^{q_{\delta^+}} q^2 \ln (q + 1)\, dq = \frac{1}{3} \left[(q_{\delta^+}^{3} + 1) \ln (q_{\delta^+} + 1) - \frac{1}{6} (2q_{\delta^+}^{3} - 3q_{\delta^+}^{2} \right.$$

$$\left. + 6q_{\delta^+} - 5 + 12 \ln 2) \right]$$

$$\int_1^{q_{\delta^+}} q^3 \ln (q + 1)\, dq = \frac{1}{4}\Big[(q_{\delta^+}^4 - 1) \ln (q_{\delta^+} + 1) - \frac{1}{12} (3q_{\delta^+}^4 - 4q_{\delta^+}^3$$

$$+ 6q_{\delta^+}^2 - 12q_{\delta^+} + 7)\Big]$$

$$\int_1^{q_{\delta^+}} q^4 \ln (q + 1)\, dq = \frac{1}{5}\Big[(q_{\delta^+}^5 + 1) \ln (q_{\delta^+} + 1) - \frac{1}{60} (12q_{\delta^+}^5 - 15q_{\delta^+}^4$$

$$+ 20q_{\delta^+}^3 - 30q_{\delta^+}^2 + 60q_{\delta^+} - 47 + 120 \ln 2)\Big]$$

$$\int_1^{q_{\delta^+}} q^5 \ln (q + 1)\, dq = \frac{1}{6}\Big[(q_{\delta^+}^6 - 1) \ln (q_{\delta^+} + 1) - \frac{1}{60} (10q_{\delta^+}^6$$

$$- 12q_{\delta^+}^5 + 15q_{\delta^+}^4 - 20q_{\delta^+}^3 + 30q_{\delta^+}^2 - 60q_{\delta^+} + 37)\Big]$$

$$\int_1^{q_{\delta^+}} q^6 \ln (q + 1)\, dq = \frac{1}{7}\Big[(q_{\delta^+}^7 + 1) \ln (q_{\delta^+} + 1) - \frac{1}{420} (60q_{\delta^+}^7 - 70q_{\delta^+}^6$$

$$+ 84q_{\delta^+}^5 - 105q_{\delta^+}^4 + 140q_{\delta^+}^3 - 210q_{\delta^+}^2 + 420q_{\delta^+}$$

$$- 319 + 840 \ln 2)\Big]$$

$$\int_1^{q_{\delta^+}} q^7 \ln (q + 1)\, dy = \frac{1}{8}\Big[(q_{\delta^+}^8 - 1) \ln (q_{\delta^+} + 1) - \frac{1}{840} (105q_{\delta^+}^8 - 120q_{\delta^+}^7$$

$$+ 140q_{\delta^+}^6 - 168q_{\delta^+}^5 + 210q_{\delta^+}^4 - 280q_{\delta^+}^3 + 420q_{\delta^+}^2$$

$$- 840q_{\delta^+} + 533)\Big]$$

$$\int_1^{q_{\delta^+}} q^8 \ln (q + 1)\, dq = \frac{1}{9}\Big[(q_{\delta^+}^{9} + 1) \ln (q_{\delta^+} + 1) - 2 \ln 2$$

$$- \frac{1}{840} (\frac{840}{9} q_{\delta^+}^{9} - 105\, q_{\delta^+}^{8} + 120 q_{\delta^+}^{7} - 140 q_{\delta^+}^{6}$$

$$+ 168 q_{\delta^+}^{5} - 210 q_{\delta^+}^{4} + 280 q_{\delta^+}^{3} - 420 q_{\delta^+}^{2}$$

$$+ 840 q_{\delta^+} - \frac{840}{9} - 533)\Big]$$

$$\int_1^{q_{\delta^+}} q^9 \ln (q + 1)\, dq = \frac{1}{10} \Big[(q_{\delta^+}^{10} - 1) \ln (q_{\delta^+} + 1) - \frac{1}{840} (84 q_{\delta^+}^{10}$$

$$- \frac{840}{9} q_{\delta^+}^{9} + 105 q_{\delta^+}^{8} - 120 q_{\delta^+}^{7} + 140 q_{\delta^+}^{6} - 168 q_{\delta^+}^{5}$$

$$+ 210 q_{\delta^+}^{4} - 280 q_{\delta^+}^{3} + 420 q_{\delta^+}^{2} - 840 q_{\delta^+} + \frac{840}{9} + 449)\Big]$$

$$\int_1^{q_{\delta^+}} q^{10} \ln (q + 1)\, dq = \frac{1}{11}\Big[(q_{\delta^+}^{11} + 1) \ln (q_{\delta^+} + 1) - 2 \ln 2$$

$$- \frac{1}{840} (\frac{840}{11} q_{\delta^+}^{11} - 84 q_{\delta^+}^{10} + \frac{840}{9} q_{\delta^+}^{9} - 105 q_{\delta^+}^{8}$$

$$+ 120 q_{\delta^+}^{7} - 140 q_{\delta^+}^{6} + 168 q_{\delta^+}^{5} - 210 q_{\delta^+}^{4} + 280 q_{\delta^+}^{3}$$

$$- 420 q_{\delta^+}^{2} + 840 q_{\delta^+} + \frac{840}{11} - \frac{840}{9} - 449)\Big]$$

$$\int_1^{q_{\delta^+}} q^{11} \ln (q + 1)\, dq = \frac{1}{12}\Big[(q_{\delta^+}^{12} - 1) \ln (q_{\delta^+} + 1) - \frac{1}{840} (70q_{\delta^+}^{12}$$

$$- \frac{840}{11} q_{\delta^+}^{11} + 84q_{\delta^+}^{10} - \frac{840}{9} q_{\delta^+}^{9} + 105q_{\delta^+}^{8} - 120q_{\delta^+}^{7}$$

$$+ 140q_{\delta^+}^{6} - 168q_{\delta^+}^{5} + 210q_{\delta^+}^{4} - 280q_{\delta^+}^{3} + 420q_{\delta^+}^{2}$$

$$- 840q_{\delta^+} + 840 \left(\frac{1}{11} + \frac{1}{9}\right) + 379)\Big]$$

$$\int_0^{\delta^+} \frac{q}{y^+} (q^3 - 1)\, dy^+ = 2 \ln (q_{\delta^+} + 1) - 2 \ln 2 + \frac{1}{2} q_{\delta^+}^{4} + q_{\delta^+}^{2} - 2q_{\delta^+} + \frac{1}{2}$$

$$\int_0^{\delta^+} \frac{q}{y^+} (2q^3 - 3q^2 + 1)\, dy^+ = 4 \ln (q_{\delta^+} + 1) - 4 \ln 2 + q_{\delta^+}^{4} - 2q_{\delta^+}^{3}$$

$$+ 2q_{\delta^+}^{2} - 4q_{\delta^+} + 3$$

$$\int_0^{\delta^+} \frac{q}{y^+} (3q^5 - 5q^3 + 2)\, dy^+ = -4 \ln (q_{\delta^+} + 1) + 4 \ln 2 + q_{\delta^+}^{6} - q_{\delta^+}^{4}$$

$$- 2q_{\delta^+}^{2} + 4q_{\delta^+} - 2$$

$$\int_0^{\delta^+} \frac{q}{y^+} (12q^5 - 15q^4 - 20q^3 + 30q^2 - 7)\, dy^+ = - 16 \ln (q_{\delta^+} + 1)$$

$$+ 16 \ln 2 + 4q_{\delta^+}^{6} - 6q_{\delta^+}^{5} - 4q_{\delta^+}^{4} + 10q_{\delta^+}^{3}$$

$$- 8q_{\delta^+}^{2} + 16q_{\delta^+} - 12$$

$$\int_0^{\delta^+} \frac{q}{y^+} (q^6 - 3q^4 + 3q^2 - 1)\, dy^+ = \frac{2}{7} q_{\delta^+}^7 - \frac{4}{5} q_{\delta^+}^5 + \frac{2}{3} q_{\delta^+}^3 - \frac{16}{105}$$

$$\int_0^{\delta^+} \frac{q}{y^+} (15q^7 - 42q^5 + 35q^3 - 8)\, dy^+ = 16 \ln (q_{\delta^+} + 1) - 16 \ln 2$$

$$+ \frac{15}{4} q_{\delta^+}^8 - 9q_{\delta^+}^6 + 4q_{\delta^+}^4 + 8q_{\delta^+}^2 - 16q_{\delta^+} + \frac{37}{4}$$

$$\int_0^{\delta^+} \frac{q}{y^+} (630q^{11} - 3080q^9 + 5940q^7 - 5544q^5 + 2310q^3 - 256)\, dy^+$$

$$= 512 \ln (q_{\delta^+} + 1) - 512 \ln 2 + 105\, q_{\delta^+}^{12} - 490q_{\delta^+}^{10}$$

$$+ \frac{1745}{2} q_{\delta^+}^8 - \frac{2054}{3} q_{\delta^+}^6 + 128q_{\delta^+}^4 + 256q_{\delta^+}^2 - 512q_{\delta^+} + \frac{1951}{6}$$

$$\int_0^{\delta^+} \frac{q}{y^+} (70q^9 - 270q^7 + 378q^5 - 210q^3 + 32)\, dy^+ = - 64 \ln (q_{\delta^+} + 1)$$

$$+ 64 \ln 2 + 14q_{\delta^+}^{10} - 50q_{\delta^+}^8 + \frac{178}{3} q_{\delta^+}^6 - 16q_{\delta^+}^4 - 32q_{\delta^+}^2$$

$$+ 64q_{\delta^+} - \frac{118}{3}$$

$$\int_0^{\delta^+} \frac{q}{y^+} (5q^7 - 7q^5 + 2)\, dy^+ = - 4 \ln (q_{\delta^+} + 1) + 4 \ln 2 + \frac{5}{4} q_{\delta^+}^8$$

$$- \frac{2}{3} q_{\delta^+}^6 - q_{\delta^+}^4 - 2q_{\delta^+}^2 + 4q_{\delta^+} - \frac{19}{12}$$

$$\int_0^{\delta^+} \frac{q}{y^+}(7q^9 - 9q^7 + 2)\, dy^+ = -4\ln(q_{\delta^+} + 1) + 4\ln 2 + \frac{7}{5} q_{\delta^+}^{10}$$

$$- \frac{1}{2} q_{\delta^+}^{8} - \frac{2}{3} q_{\delta^+}^{6} - q_{\delta^+}^{4} - 2q_{\delta^+}^{2} + 4q_{\delta^+} - \frac{37}{30}$$

$$\int_0^{\delta^+} \frac{q}{y^+}(9q^{11} - 11q^9 + 2)\, dy^+ = -4\ln(q_{\delta^+} + 1) + 4\ln 2 + \frac{3}{2} q_{\delta^+}^{12}$$

$$- \frac{2}{5} q_{\delta^+}^{10} - \frac{1}{2} q_{\delta^+}^{8} - \frac{2}{3} q_{\delta^+}^{6} - q_{\delta^+}^{4} - 2q_{\delta^+}^{2} + 4q_{\delta^+} - \frac{14}{15}$$

APPENDIX C

Digital Computations for the Coefficients to the Governing Equations

Before evaluating the coefficients define the following terms.

$$p_2 = q^2 - 1 \qquad p_{2_\delta} = q^2_{\delta^+} - 1 = \alpha_s \delta^+$$

$$p_3 = q^3 - 1 \qquad p_3^* = 2q^3 - 3q^2 + 1$$

$$p_5 = 3q^5 - 5q^3 + 2 \qquad p_5^* = 12q^5 - 15q^4 - 20q^3 + 30q^2 - 7$$

$$p_7 = 5q^7 - 7q^5 + 2 \qquad p_7^* = 15q^7 - 42q^5 + 35q^3 - 8$$

$$p_8 = 3q^8 - 4q^6 + 1 \qquad p_9 = 7q^9 - 9q^7 + 2$$

$$p_9^* = 70q^9 - 270q^7 + 378q^5 - 210q^3 + 32$$

$$p_{10} = 4q^{10} - 5q^8 + 1$$

$$p_{11} = 9q^{11} - 11q^9 + 2$$

$$p_{11}^* = 630q^{11} - 3080q^9 + 5940q^7 - 5544q^5 + 2310q^3 - 256$$

$$p_{12} = 5q^{12} - 6q^{10} + 1$$

$$\frac{1}{\alpha_s^2} k_1 = \int_0^{\delta^+} y^+ q \, dy^+ = \frac{2}{15\,\alpha_s^2} p_{5_\delta}$$

$$\frac{1}{\alpha_s^3} k_2 = \int_0^{\delta^+} y^{+^2} q \, dy^+ = \frac{2}{105 \, \alpha_s^3} p^*_{7_\delta}$$

$$\frac{1}{\alpha_s^4} k_3 = \int_0^{\delta^+} y^{+^3} q \, dy^+ = \frac{1}{315 \, \alpha_s^4} p^*_{9_\delta}$$

$$\frac{1}{\alpha_s^5} k_4 = \int_0^{\delta^+} y^{+^4} q \, dy^+ = \frac{1}{3465 \, \alpha_s^5} p^*_{11_\delta}$$

$$\frac{1}{\alpha_s} l_i = \int_0^{\delta^+} q^i \, dy^+ = \frac{2}{(i+2) \, \alpha_s} \left(q_{\delta^+}^{i+2} - 1\right) \qquad i = 1,4,6$$

$$\overline{p}_3 = \int_0^{\delta^+} \frac{q}{y^+} p_3 \, dy^+ = 2 \ln \left(q_{\delta^+} + 1\right) - 2 \ln 2 + \frac{1}{2} q_{\delta^+}^4 + q_{\delta^+}^2 - 2q_{\delta^+} + \frac{1}{2}$$

$$\overline{p}^*_3 = \int_0^{\delta^+} \frac{q}{y^+} p^*_3 \, dy^+ = 4 \ln \left(q_{\delta^+} + 1\right) - 4 \ln 2 + q_{\delta^+}^4 - 2q_{\delta^+}^3 + 2q_{\delta^+}^2$$
$$- 4q_{\delta^+} + 3$$

$$\overline{p}_5 = \int_0^{\delta} \frac{q}{y^+} p_5 \, dy^+ = - 4 \ln \left(q_{\delta^+} + 1\right) + 4 \ln 2 + q_{\delta^+}^6 - q_{\delta^+}^4 - 2q_{\delta^+}^2$$
$$+ 4q_{\delta^+} - 2$$

$$\overline{p_5^*} = \int_0^{\delta^+} \frac{q}{y^+} p_5^* \, dy^+ = -16 \ln (q_{\delta^+} + 1) + 16 \ln 2 + 4q_{\delta^+}^6 - 6q_{\delta^+}^5 - 4q_{\delta^+}^4 + 10q_{\delta^+}^3 - 8q_{\delta^+}^2 + 16q_{\delta^+} - 12$$

$$\overline{p_7} = \int_0^{\delta^+} \frac{q}{y^+} p_7 \, dy^+ = -4 \ln (q_{\delta^+} + 1) + 4 \ln 2 + \frac{5}{4} q_{\delta^+}^8 - \frac{2}{3} q_{\delta^+}^6 - q_{\delta^+}^4 - 2q_{\delta^+}^2 + 4q_{\delta^+} - \frac{19}{12}$$

$$\overline{p_7^*} = \int_0^{\delta^+} \frac{q}{y^+} p_7^* \, dy^+ = 16 \ln (q_{\delta^+} + 1) - 16 \ln 2 + \frac{15}{4} q_{\delta^+}^8 - 9q_{\delta^+}^6 + 4q_{\delta^+}^4 + 8q_{\delta^+}^2 - 16q_{\delta^+} + \frac{37}{4}$$

$$\overline{p_9} = \int_0^{\delta^+} \frac{q}{y^+} p_9 \, dy^+ = -4 \ln (q_{\delta^+} + 1) + 4 \ln 2 + \frac{7}{5} q_{\delta^+}^{10} - \frac{1}{2} q_{\delta^+}^8 - \frac{2}{3} q_{\delta^+}^6 - q_{\delta^+}^4 - 2q_{\delta^+}^2 + 4q_{\delta^+} - \frac{37}{30}$$

$$\overline{p_9^*} = \int_0^{\delta^+} \frac{q}{y^+} p_9^* \, dy^+ = -64 \ln (q_{\delta^+} + 1) + 64 \ln 2 + 14q_{\delta^+}^{10} - 50q_{\delta^+}^8 - \frac{178}{3} q_{\delta^+}^6 - 16q_{\delta^+}^4 - 32q_{\delta^+}^2 + 64q_{\delta^+} - \frac{118}{3}$$

$$\overline{p_{11}} = \int_0^{\delta^+} \frac{q}{y^+} p_{11} \, dy^+ = -4 \ln (q_{\delta^+} + 1) + 4 \ln 2 + \frac{3}{2} q_{\delta^+}^{12} - \frac{2}{5} q_{\delta^+}^{10} - \frac{1}{2} q_{\delta^+}^8 - \frac{2}{3} q_{\delta^+}^6 - q_{\delta^+}^4 - 2q_{\delta^+}^2 + 4q_{\delta^+} - \frac{14}{15}$$

$$\overline{p}_{11}^{*} = \int_{0}^{\delta^+} \frac{q}{y^+} p_{11}^{*} \, dy^+ = 512 \ln (q_{\delta^+} + 1) - 512 \ln 2 + 105 q_{\delta^+}^{12} - 490 q_{\delta^+}^{10}$$

$$+ \frac{1745}{2} q_{\delta^+}^{8} - \frac{2054}{3} q_{\delta}^{6} + 128 q_{\delta^+}^{4} + 256 q_{\delta^+}^{2}$$

$$- 512 q_{\delta^+} + \frac{1951}{6}$$

$$\frac{1}{\alpha_s^2} m_3 = \int_{0}^{\delta^+} y^+ q^3 \, dy^+ = \frac{2}{35 \alpha_s^2} p_{7_\delta}$$

$$\frac{1}{\alpha_s^2} m_4 = \int_{0}^{\delta^+} y^+ q^4 \, dy^+ = \frac{1}{12 \alpha_s^2} p_{8_\delta}$$

$$\frac{1}{\alpha_s^2} m_5 = \int_{0}^{\delta^+} y^+ q^5 \, dy^+ = \frac{2}{63 \alpha_s^2} p_{9_\delta}$$

$$\frac{1}{\alpha_s^2} m_6 = \int_{0}^{\delta^+} y^+ q^6 \, dy^+ = \frac{1}{20 \alpha_s^2} p_{10_\delta}$$

$$\frac{1}{\alpha_s^2} m_7 = \int_{0}^{\delta^+} y^+ q^7 \, dy^+ = \frac{2}{99 \alpha_s^2} p_{11_\delta}$$

$$\frac{1}{\alpha_s^2} m_8 = \int_{0}^{\delta^+} y^+ q^8 \, dy^+ = \frac{1}{30 \alpha_s^2} p_{12_\delta}$$

$$\frac{1}{\kappa\alpha_s^2} n_1 = \int_0^{\delta^+} y^+ q\, u^+ \, dy^+ = \frac{2}{15\ \kappa\alpha_s^2} [\kappa\, u_e^+ \, p_{5_\delta} - \bar{p}_5]$$

$$\frac{1}{\kappa\alpha_s^3} n_2 = \int_0^{\delta^+} y^{+^2} q\, u^+ \, dy^+ = \frac{2}{105\ \kappa\alpha_s^3} [\kappa\, u_e^+ \, p^*_{7_\delta} - \bar{p}^*_7]$$

$$\frac{1}{\kappa\alpha_s^4} n_3 = \int_0^{\delta^+} y^{+^3} q\, u^+ \, dy^+ = \frac{1}{315\ \kappa\alpha_s^4} [\kappa\, u_e^+ \, p^*_{9_\delta} - \bar{p}^*_9]$$

$$\frac{1}{\kappa\alpha_s^5} n_4 = \int_0^{\delta^+} y^{+^4} q\, u^+ \, dy^+ = \frac{1}{3465\ \kappa\alpha_s^5} [\kappa\, u_e^+ \, p^*_{11_\delta} - \bar{p}^*_{11}]$$

$$\frac{1}{\kappa\alpha_s^2} K_1 = \int_0^{\delta^+} y^+ u^+ \, dy^+ = \frac{1}{2\ \kappa\alpha_s^2} [\alpha_s^2 \, \delta^{+^2} \, \kappa\, u_e^+ - k_1]$$

$$\frac{1}{\kappa\alpha_s^3} K_2 = \int_0^{\delta^+} y^{+^2} u^+ \, dy^+ = \frac{1}{3\ \kappa\alpha_s^3} [\alpha_s^3 \, \delta^{+^3} \, \kappa\, u_e^+ - k_2]$$

$$\frac{1}{\kappa\alpha_s^4} K_3 = \int_0^{\delta^+} y^{+^3} u^+ \, dy^+ = \frac{1}{4\ \kappa\alpha_s^4} [\alpha_s^4 \, \delta^{+^4} \, \kappa\, u_e^+ - k_3]$$

$$\frac{1}{\kappa\alpha_s^5} K_4 = \int_0^{\delta^+} y^{+^4} u^+ \, dy^+ = \frac{1}{5\ \kappa\alpha_s^5} [\alpha_s^5 \, \delta^{+^5} \, \kappa\, u_e^+ - k_4]$$

$$\frac{1}{\kappa\alpha_s^2} L_3 = \int_0^{\delta^+} y^+ q^3 u^+ dy^+ = \frac{1}{\kappa\alpha_s^2} [\kappa u_e^+ m_3 - \frac{2}{35} \bar{p}_7]$$

$$\frac{1}{\kappa\alpha_s^2} L_5 = \int_0^{\delta^+} y^+ q^5 u^+ dy^+ = \frac{1}{\kappa\alpha_s^2} [\kappa u_e^+ m_5 - \frac{2}{63} \bar{p}_9]$$

$$\frac{1}{\kappa\alpha_s^2} L_7 = \int_0^{\delta^+} y^+ q^7 u^+ dy^+ = \frac{1}{\kappa\alpha_s^2} [\kappa u_e^+ m_7 - \frac{2}{99} \bar{p}_{11})$$

$$\frac{1}{\kappa^2\alpha_s^2} M_1 = \int_0^{\delta^+} y^+ u^{+2} dy^+ = \frac{1}{2\kappa^2\alpha_s^2} [\alpha_s^2 \delta^{+2} \kappa^2 u_e^{+2} - 2 n_1]$$

$$\frac{1}{\kappa^2\alpha_s^3} M_2 = \int_0^{\delta^+} y^{+2} u^{+2} dy^+ = \frac{1}{3\kappa^2\alpha_s^3} [\alpha_s^3 \delta^{+3} \kappa^2 u_e^{+2} - 2 n_2]$$

$$\frac{1}{\kappa^2\alpha_s^4} M_3 = \int_0^{\delta^+} y^{+3} u^{+2} dy^+ = \frac{1}{4\kappa^2\alpha_s^4} [\alpha_s^4 \delta^{+4} \kappa^2 u_e^{+2} - 2 n_3]$$

$$\frac{1}{\kappa^2\alpha_s^5} M_4 = \int_0^{\delta^+} y^{+4} u^{+2} dy^+ = \frac{1}{5\kappa^2\alpha_s^5} [\alpha_s^5 \delta^{+5} \kappa^2 u_e^{+2} - 2 n_4]$$

$$\frac{1}{\kappa\alpha_s} N_1 = \int_0^{\delta^+} q u^+ dy^+ = \frac{2}{3\kappa\alpha_s} [\kappa u_e^+ p_{3_\delta} - \bar{p}_3]$$

$$\frac{1}{\kappa\alpha_s} N_3 = \int_0^{\delta^+} q^3\, u^+\, dy^+ = \frac{1}{\kappa\alpha_s} [\alpha_s\, \delta^+\, \kappa\, u_e^+\, q^3_{\delta^+} - l_4 - \frac{3}{2} n_1]$$

$$\frac{1}{\kappa\alpha_s} N_5 = \int_0^{\delta^+} q^5\, u^+\, dy^+ = \frac{1}{\kappa\alpha_s} [\alpha_s\, \delta^+\, \kappa\, u_e^+\, q^5_{\delta^+} - l_6 - \frac{5}{2} L_3]$$

$$\frac{1}{\kappa\alpha_s} N_7 = \int_0^{\delta^+} q^7\, u^+\, dy^+ = \frac{1}{\kappa\alpha_s} [\alpha_s\, \delta^+\, \kappa\, u_e^+\, q^7_{\delta^+} - l_8 - \frac{7}{2} L_5]$$

$$\frac{1}{\kappa\alpha_s} U_1 = \int_0^{\delta^+} u^+\, dy^+ = \frac{1}{\kappa\alpha_s} [\alpha_s\, \delta^+\, \kappa\, u_e^+ - l_1] = I_0^{(e)}$$

$$\frac{1}{\kappa^2\alpha_s} U_2 = \int_0^{\delta^+} u^{+2}\, dy^+ = \frac{1}{\kappa^2\alpha_s} [\alpha_s\, \delta^+\, \kappa^2\, u_e^{+2} - 2\, N_1]$$

Evaluation of the coefficients to the governing equations is given below.

$$I_0 = \int_0^{y^+} u^+\, dy^+ = y^+\, u^+ - \frac{2}{3\,\kappa\alpha_s} P_3$$

$$I_2 = \int_0^{y^+} \frac{\partial u^+}{\partial \alpha_s}\, dy^+ = \frac{2}{3\,\kappa\alpha_s^2} P_3 - \frac{1}{\kappa\alpha_s} y^+$$

$$J_0 = \int_0^{y^+} w^+ \, dy^+ = \sigma \, u_e^+ \, y^+ + \Theta \left[y^+ u^+ - \frac{2}{3 \, \kappa\alpha_s} p_3 \right]$$

$$+ \frac{2(\beta-\sigma-\Theta)}{\delta^+} \left[\frac{1}{2} y^{+^2} u^+ - \frac{1}{15 \, \kappa\alpha_s^2} p_5 \right]$$

$$+ \frac{(\Theta-\beta+\sigma)}{\delta^{+^2}} \left[\frac{1}{3} y^{+^3} u^+ - \frac{2}{315 \, \kappa\alpha_s^3} p_7^* \right]$$

$$J_2 = \int_0^{y^+} \frac{\partial w^+}{\partial \alpha_s} dy^+ = \frac{1}{\kappa\alpha_s} \left\{ \Theta \left[\frac{2}{3 \, \alpha_s} p_3 - y^+ \right] \right.$$

$$+ \frac{2(\beta-\sigma-\Theta)}{\delta^+} \left[\frac{2}{15 \, \alpha_s^2} p_5 - \frac{1}{2} y^{+^2} \right]$$

$$\left. + \frac{(\Theta-\beta-\sigma)}{\delta^{+^2}} \left[\frac{2}{105 \, \alpha_s^3} p_7^* - \frac{1}{3} y^{+^3} \right] \right\}$$

$$J_3 = \int_0^{y^+} \frac{\partial w^+}{\partial \delta^+} dy^+ = - \frac{2(\beta-\sigma-\Theta)}{\delta^{+^2}} \left[\frac{1}{2} y^{+^2} u^+ - \frac{1}{15 \, \kappa\alpha_s^2} p_5 \right]$$

$$- \frac{2(\Theta-\beta+\sigma)}{\delta^{+^3}} \left[\frac{1}{3} y^{+^3} u^+ - \frac{2}{315 \, \kappa\alpha_s^3} p_7^* \right]$$

$$J_4 = \int_0^{y^+} \frac{\partial w^+}{\partial \Theta} dy^+ = y^+ u^+ - \frac{2}{3 \, \kappa\alpha_s} p_3$$

$$- \frac{2}{\delta^+} \left[\frac{1}{2} y^{+^2} u^+ - \frac{1}{15 \, \kappa\alpha_s^2} p_5 \right]$$

$$+ \frac{1}{\delta^{+2}} \left[\frac{1}{3} y^{+3} u^{+} - \frac{2}{315 \ \kappa \alpha_s^3} p_7^* \right]$$

$$J_5 = \int_0^{y^+} \frac{\partial w^+}{\partial \beta} dy^+ = \frac{2}{\delta^+} \left[\frac{1}{2} y^{+2} u^+ - \frac{1}{15 \ \kappa \alpha_s^2} p_5 \right]$$

$$- \frac{1}{\delta^{+2}} \left[\frac{1}{3} y^{+3} u^+ - \frac{2}{315 \ \kappa \alpha_s^3} p_7^* \right]$$

$$J_6 = \int_0^{y^+} \frac{\partial w^+}{\partial \sigma} dy^+ = u_e^+ y^+ - \frac{2}{\delta^+} \left[\frac{1}{2} y^{+2} u^+ - \frac{1}{15 \ \kappa \alpha_s^2} p_5 \right]$$

$$+ \frac{1}{\delta^{+2}} \left[\frac{1}{3} y^{+3} u^+ - \frac{2}{315 \ \kappa \alpha_s^3} p_7^* \right]$$

$$G_1 = \int_0^{\delta^+} u^{+2} \, dy^+ = \frac{1}{\kappa^2 \alpha_s} U_2$$

$$G_2 = \int_0^{\delta^+} \left(u^+ \frac{\partial u^+}{\partial \alpha_s} - \frac{\partial u^+}{\partial y^+} I_2 \right) dy^+ = \frac{1}{3 \ \kappa^2 \alpha_s^2} \left[\kappa \ u_e^+ p_{3_\delta}^* - 4\bar{p}_3 + 4 p_{3_\delta} \right]$$

$$G_3 = \int_0^{\delta^+} u^+ w^+ dy^+ = \frac{1}{\kappa^2 \alpha_s} \left[\sigma \kappa \ u_e^+ U_1 + \theta \ U_2 \right.$$

$$\left. + \frac{2(\beta - \sigma - \theta)}{\alpha_s \delta^+} M_1 + \frac{(\theta - \beta + \sigma)}{\alpha_s^2 \delta^{+2}} M_2 \right]$$

$$G_4 = \int_0^{\delta^+} \left[w^+ \frac{\partial u^+}{\partial \alpha_s} - \frac{\partial u^+}{\partial y^+} J_2 + \frac{\delta^+(q_{\delta^+}-1)}{\alpha_s q_{\delta^+}} \frac{\partial u^+}{\partial y^+} J_3 \right] dy^+$$

$$= \frac{1}{\kappa^2 \alpha_s^2} \left\{ \frac{1}{3} \kappa\, u_e^+ \sigma\, p^*_{3_\delta} + \frac{1}{3} \Theta \left(\kappa\, u_e^+ p^*_{3_\delta} - 4\, \overline{p}_3 + 4 p_{3_\delta}\right) \right.$$

$$+ \frac{2(\beta-\sigma-\Theta)}{30\, \alpha_s \delta^+} \left(\kappa\, u_e^+ p^*_{5_\delta} - 8\overline{p}_5 + 4 p_{5_\delta}\right)$$

$$\left. + \frac{(\Theta-\beta+\sigma)}{\alpha_s^2 \delta^{+2}} \left[\kappa\, u_e^+ \left(\frac{2}{105} p^*_{7_\delta} - \frac{1}{3} \alpha_s^3 \delta^{+3}\right) - \frac{4}{105} \overline{p}^*_7 + \frac{2}{3} k_2 \right] \right\}$$

$$- \frac{(q_{\delta^+}-1)}{\kappa^2 \alpha_s^2 q_{\delta^+}} \left[\frac{2(\beta-\sigma-\Theta)}{15\, \alpha_s \delta^+} \left(\kappa\, u_e^+ p_{5_\delta} - 2\, \overline{p}_5\right) \right.$$

$$\left. + \frac{4(\Theta-\beta+\sigma)}{315\, \alpha_s^2 \delta^{+2}} \left(\kappa\, u_e^+ p^*_{7_\delta} - 2\, \overline{p}^*_7\right) \right]$$

$$G_5 = \frac{\kappa \delta^+}{q_{\delta^+}} \int_0^{\delta^+} \frac{\partial u^+}{\partial y^+} J_3 \; dy^+ = - \frac{1}{\kappa \alpha_s q_{\delta^+}} \left[\frac{2(\beta-\sigma-\Theta)}{15\, \alpha_s \delta^+} \left(\kappa\, u_e^+ p_{5_\delta} - 2\, \overline{p}_5\right) \right.$$

$$\left. + \frac{4(\Theta-\beta+\sigma)}{315\, \alpha_s^2 \delta^{+2}} \left(\kappa\, u_e^+ p^*_{7_\delta} - 2\, \overline{p}^*_7\right) \right]$$

$$G_6 = \int_0^{\delta^+} \frac{\partial u^+}{\partial y^+} J_4 \; dy^+ = \frac{2}{3\, \kappa^2 \alpha_s} \left[\kappa\, u_e^+ p_{3_\delta} - 2\, \overline{p}_3 \right] - G_7$$

$$G_7 = \int_0^{\delta^+} \frac{\partial u^+}{\partial y^+} J_5 \; dy^+ = \frac{1}{\kappa^2 \alpha_s^2 \delta^+} \left[n_1 - \frac{2}{15} \overline{p}_5 - \frac{1}{\alpha_s \delta^+} \left(\frac{1}{3} n_2 - \frac{2}{315} \overline{p}^*_7 \right) \right]$$

$$G_8 = \int_0^{\delta^+} \frac{\partial u^+}{\partial y^+} J_6 \, dy^+ = \frac{\kappa u_e^+}{\kappa^2 \alpha_s} I_1 - G_7$$

$$G_9 = \int_0^{\delta^+} \left[w^{+2} + \frac{\partial u^+}{\partial y^+} I_0 \right] dy^+ = \frac{1}{\kappa^2 \alpha_s} \left[\sigma^2 (\kappa u_e^+)^2 \alpha_s \delta^+ \right.$$

$$+ \Theta (2 \sigma \kappa u_e^+ U_1 + \Theta U_2) + \frac{2(\beta-\sigma-\Theta)}{\alpha_s \delta^+} (2 \sigma \kappa u_e^+ K_1$$

$$+ 2 \Theta M_1 + \frac{2(\beta-\sigma-\Theta)}{\alpha_s \delta^+} M_2) + \frac{(\Theta-\beta+\sigma)}{\alpha_s^2 \delta^{+2}} (2 \sigma \kappa u_e^+ K_2$$

$$+ 2 \Theta M_2 + \frac{4(\beta-\sigma-\Theta)}{\alpha_s \delta^+} M_3 + \frac{(\Theta-\beta+\sigma)}{\alpha_s^2 \delta^{+2}} M_4)$$

$$\left. + \frac{2}{3} (\kappa u_e^+ p_{3_\delta} - 2 \bar{p}_3) \right]$$

$$G_{10} = \int_0^{\delta^+} \left[u^+ \frac{\partial w^+}{\partial \alpha_s} - \frac{\partial w^+}{\partial y^+} I_2 - \frac{\delta^+ (q_{\delta^+} - 1)}{\alpha_s q_{\delta^+}} u^+ \frac{\partial w^+}{\partial \delta^+} \right] dy^+$$

$$= \frac{1}{\kappa^2 \alpha_s^2} \left\{ \frac{1}{3} \Theta (\kappa u_e^+ p^*_{3_\delta} - 4 \bar{p}_3 + 4 p_{3_\delta}) \right.$$

$$+ \frac{2(\beta-\sigma-\Theta)}{\alpha_s \delta^+} \left(2 n_1 - \frac{2}{3} \alpha_s \delta^+ \kappa u_e^+ p_{3_\delta} + k_1 + \frac{q_{\delta^+} - 1}{q_{\delta^+}} M_1 \right)$$

$$+ \frac{(\Theta-\beta+\sigma)}{\alpha_s^2 \delta^{+2}} \left[n_2 - \frac{2}{3} (m_4 + 2 L_3 - \alpha_s^2 \delta^{+2} \kappa u_e^+) \right.$$

$$+ \frac{1}{3} (2 k_2 + \alpha_s^3 \delta^{+3} \kappa u_e^+) + \frac{2(q_{\delta^+}-)}{q_{\delta^+}} M_2 \Big] \Big\}$$

$$G_{11} = \frac{\kappa \delta^+}{q_{\delta^+}} \int_0^{\delta^+} u^+ \frac{\partial w^+}{\partial \delta^+} dy^+ = - \frac{2}{\kappa \alpha_s q_{\delta^+}} \frac{(\beta-\sigma-\Theta)}{\alpha_s \delta^+} \left[M_1 - \frac{1}{\alpha_s \delta^+} M_2 \right]$$

$$G_{12} = \int_0^{\delta^+} (w^+)^2 dy^+ = G_9 - \frac{2}{3 \kappa^2 \alpha_s} (\kappa u_e^+ p_{3_\delta} - 2 \bar{p}_3)$$

$$G_{13}^{(a)} = \int_0^{\delta^+} (w^+ \frac{\partial w^+}{\partial \alpha_s} - \frac{\partial w^+}{\partial y^+} J_2) dy^+$$

$$= \frac{\Theta}{3 \kappa^2 \alpha_s^2} \left[(\sigma + \Theta) \kappa u_e^+ p_{3_\delta}^* + 4 \Theta (p_{3_\delta} - \bar{p}_3) \right]$$

$$+ \frac{(\Theta-\beta+\sigma)}{\kappa^2 \alpha_s^2 p_{2_\delta}} \left\{ \sigma \kappa u_e^+ \left[- \frac{1}{15} p_{5_\delta}^* + \frac{1}{p_{2_\delta}} (k_2 - \frac{1}{3} p_{2_\delta}^3) \right] \right.$$

$$- 2 \Theta \left[\kappa u_e^+ (\frac{2}{5} p_{5_\delta} - \frac{1}{2} p_{2_\delta}^2 - \frac{2}{3} p_{2_\delta} p_{3_\delta}) - \frac{8}{15} \bar{p}_5 + \frac{4}{15} p_{5_\delta} \right]$$

$$+ \frac{\Theta}{p_{2_\delta}} \left[2 n_2 - \frac{2}{105} \bar{p}_7^* + \frac{4}{3} k_2 - \frac{2}{3} (m_4 - k_1) - \frac{4}{3} (L_3 - K_1) \right]$$

$$+ \frac{4(\Theta-\beta+\sigma)}{p_{2_\delta}} \left[n_2 + \frac{2}{3} k_2 - \kappa u_e^+ p_{2_\delta} (\frac{1}{6} p_{2_\delta}^2 + \frac{2}{15} p_{5_\delta}) + L_3 - n_1 \right]$$

$$- \frac{2(\Theta-\beta+\sigma)}{p_{2_\delta}^2} \left[2 n_3 + k_3 + n_1 - \kappa u_e^+ (\frac{1}{6} p_{2_\delta}^4 + \frac{4}{15} p_{2_\delta}^2 + \frac{2}{105} p_{2_\delta} p_{7_\delta}^*) \right.$$

$$+ \frac{1}{5} L_5 - \frac{2}{3} L_3 - \frac{2}{5} m_6 + \frac{2}{3} m_4 \Big]$$

$$+ \frac{(\Theta-\beta+\sigma)}{p_{2_\delta}^{3}} \Big[n_4 - \frac{1}{3} K_4 + \frac{1}{3} k_4 - \frac{2}{105} (15\, m_8 - 42\, m_6 + 35\, m_4)$$

$$- \frac{4}{105} (15\, L_7 - 42\, L_5 + 35\, L_3 - 4\, \kappa\, u_e^+\, p_{2_\delta}^{2}) \Big] \Big\}$$

$$- \frac{p_{2_\delta} (q_{\delta^+} - 1)}{\alpha_s^2 q_{\delta^+}} G_{13}^{(b)} = - \frac{\delta^+ (q_{\delta^+} - 1)}{\alpha_s q_{\delta^+}} \int_0^{\delta^+} \left(w^+ \frac{\partial w^+}{\partial \delta^+} - \frac{\partial w^+}{\partial y^+} J_3 \right) dy^+$$

$$= \frac{2(q_{\delta^+} - 1)(\Theta-\beta+\sigma)}{\kappa^2 \alpha_s^2 q_{\delta^+}\, p_{2_\delta}} \Big\{ \sigma\, \kappa\, u_e^+ (- K_1 + \frac{1}{p_{2_\delta}} K_2)$$

$$+ \Theta \Big[- M_1 + \frac{1}{2} n_1 - \frac{1}{15} \bar{p}_5 + \frac{1}{p_{2_\delta}} \Big(M_2 - \frac{1}{3} n_2 + \frac{2}{315} \bar{p}_7^* \Big) \Big]$$

$$+ \frac{(\Theta-\beta+\sigma)}{p_{2_\delta}} \Big[\frac{1}{3} p_{2_\delta}\, \kappa\, u_e^+ (p_{2_\delta}^{2}\, \kappa\, u_e^+ + \frac{2}{5} p_{5_\delta}) + n_1 - \frac{5}{3} n_2 - L_3 \Big]$$

$$+ \frac{(\Theta-\beta+\sigma)}{p_{2_\delta}^{2}} \Big[- p_{2_\delta}\, \kappa\, u_e^+ \Big(\frac{4}{315} p_{7_\delta}^* + \frac{2}{15} p_{2_\delta} + \frac{1}{3} p_{2_\delta}^{3}\, \kappa\, u_e^+ \Big)$$

$$+ \frac{2}{3} n_1 + \frac{11}{6} n_3 - \frac{2}{3} L_3 + \frac{4}{5} L_5 + \frac{1}{15} (5\, m_4 - 3\, m_6) \Big]$$

$$+ \frac{(\Theta-\beta+\sigma)}{p_{2_\delta}^{3}} \left[\frac{1}{15} \kappa\, u_e^+ p_{2_\delta}^{2} \left(\kappa\, u_e^+ p_{2_\delta}^{3} - \frac{16}{21} \right) - \frac{7}{15} n_4 \right.$$

$$\left. + \frac{2}{315} (15\, m_8 - 42\, m_6 + 35\, m_4 + 30\, L_7 - 84\, L_5 + 70\, L_3) \right] \Bigg\}$$

$$G_{13} = \int_0^{\delta^+} \left[w^+ \frac{\partial w^+}{\partial \alpha_s} - \frac{\partial w^+}{\partial y^+} J_2 - \frac{\delta^+ (q_{\delta^+} - 1)}{\alpha_s q_{\delta^+}} \left(w^+ \frac{\partial w^+}{\partial \delta^+} - \frac{\partial w^+}{\partial y^+} J_3 \right) \right] dy^+$$

$$= G_{13}^{(a)} + \left[- \frac{p_{2_\delta} (q_{\delta^+} - 1)}{\alpha_s^2 q_{\delta^+}} G_{13}^{(b)} \right]$$

$$G_{14} = \frac{\kappa \delta^+}{q_{\delta^+}} \int_0^{\delta^+} \left(w^+ \frac{\partial w^+}{\partial \delta^+} - \frac{\partial w^+}{\partial y^+} J_3 \right) dy^+$$

$$= - \frac{\kappa \alpha_s}{q_{\delta^+} - 1} \left[- \frac{p_{2_\delta} (q_{\delta^+} - 1)}{\alpha_s^2 q_{\delta^+}} G_{13}^{(b)} \right]$$

$$G_{15} = \int_0^{\delta^+} u^+ \frac{\partial w^+}{\partial \Theta} dy^+ = \frac{1}{\kappa^2 \alpha_s} \left[U_2 - \frac{2}{p_{2_\delta}} M_1 + \frac{1}{p_{2_\delta}^{2}} M_2 \right]$$

$$G_{16} = \int_0^{\delta^+} \left(w^+ \frac{\partial w^+}{\partial \Theta} - \frac{\partial w^+}{\partial y^+} J_4 \right) dy^+$$

$$= \frac{1}{\kappa^2 \alpha_s} \Bigg\{ \sigma\, \kappa\, u_e^+ \left(U_1 - \frac{2}{p_{2_\delta}} K_1 + \frac{1}{p_{2_\delta}^{2}} K_2 \right)$$

$$+ \Theta \left[\frac{1}{3} p_{2_\delta} \kappa^2 u_e^{+2} - 2 \kappa\, u_e^+ p_{3_\delta} + \frac{8}{3} \bar{p}_3 \right.$$

$$+ \frac{2}{5\, p_{2_\delta}} (\kappa\, u_e^+ \, p_{5_\delta} - \frac{4}{3} \bar{p}_5) - \frac{2}{105\, p_{2_\delta}^2} (\kappa\, u_e^+ \, p^*_{7_\delta} - \frac{4}{3} \bar{p}^*_7) \Big] \Big] \Big\}$$

$$+ \frac{2(\beta-\sigma-\Theta)}{\kappa^2 \alpha_s} \Big\{ - \frac{1}{30} p_{2_\delta} \kappa^2 u_e^{+2} + \frac{\kappa u_e^+}{3} (2\, p_{3_\delta} + 1)$$

$$+ \frac{1}{p_{2_\delta}} \Big[- 2\, n_1 + \frac{2}{15} \kappa\, u_e^+ \, (1 - p_{5_\delta}) \Big]$$

$$+ \frac{1}{p_{2_\delta}^2} \Big[\frac{2}{315} \kappa\, u_e^+ \, p^*_{7_\delta} + \frac{1}{3} L_3 - n_1 + \frac{8}{315} \kappa\, u_e^+ + \frac{11}{16} n_2 - \frac{1}{3} m_4 \Big]$$

$$+ \frac{1}{p_{2_\delta}^3} \Big[- \frac{7}{6} n_3 + \frac{1}{15} L_5 - \frac{1}{3} n_1 + \frac{1}{5} m_6 - \frac{1}{3} m_4 \Big]$$

$$+ \frac{1}{p_{2_\delta}^4} \Big[\frac{7}{30} n_4 - \frac{2}{21} L_7 + \frac{4}{15} L_5 - \frac{2}{9} L_3 - \frac{1}{21} m_8 + \frac{2}{15} m_6 - \frac{1}{9} m_4 \Big] \Big\}$$

$$G_{17} = \int_0^{\delta^+} (u^+ w^+ - \frac{\partial w^+}{\partial y^+} I_0) \quad dy^+$$

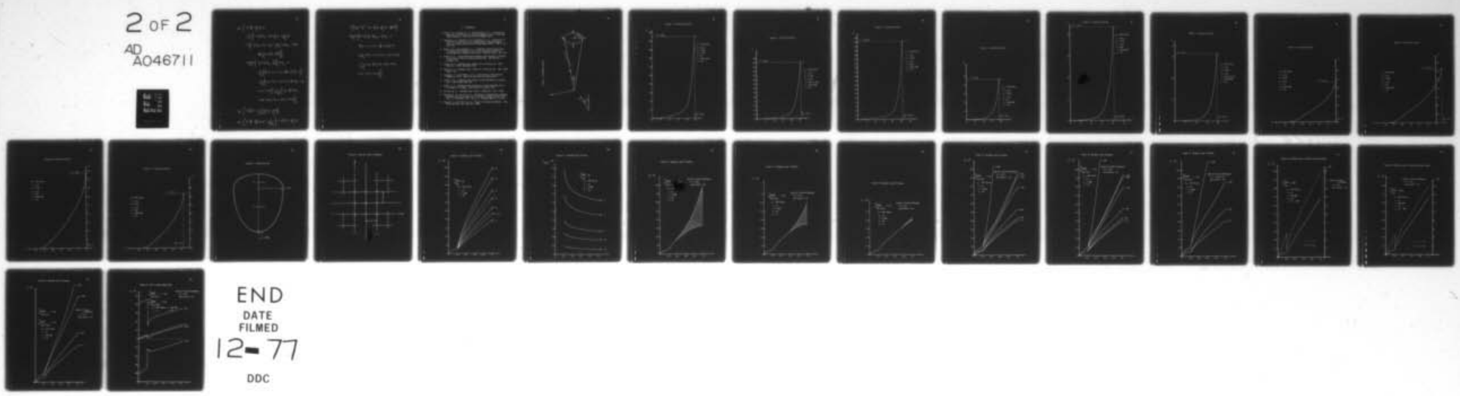
$$= G_3 - \frac{1}{\kappa^2 \alpha_s} \Big\{ \Theta\, (N_1 - \frac{2}{3} \bar{p}_3) + \frac{(\beta-\sigma-\Theta)}{3} \Big[p_{2_\delta} \kappa^2 u_e^{+2}$$

$$- \kappa\, u_e^+ \, (4\, p_{3_\delta} + 2) + \frac{6 n_1}{p_{2_\delta}} + \frac{1}{p_{2_\delta}^2} (n_2 + 2\, m_4 + 4\, L_3) \Big] \Big\}$$

$$G_{18} = \int_0^{\delta^+} u^+ \frac{\partial w^+}{\partial \sigma} dy^+ = \frac{1}{\kappa^2 \alpha_s} \Big[\kappa\, u_e^+ \, U_1 - \frac{2}{p_{2_\delta}} M_1 + \frac{1}{p_{2_\delta}^2} M_2 \Big]$$

AD-A046 711 NORTH CAROLINA STATE UNIV RALEIGH DEPT OF MECHANICAL--ETC F/G 20/4
THREE-DIMENSIONAL TURBULENT BOUNDARY LAYER ON A SPINNING CONE A--ETC(U)
OCT 77 C P FORD DAHC04-74-G-0206
UNCLASSIFIED ARO-12675.1-E NL

2 OF 2
AD A046711



END
DATE FILMED
12-77
DDC

$$G_{19} = \int_0^{\delta^+} (w^+ \frac{\partial w^+}{\partial \sigma} - \frac{\partial w^+}{\partial y^+} J_6)\, dy^+$$

$$= \frac{1}{\kappa^2 \alpha_s} \left\{ \sigma \kappa u_e^+ (\frac{1}{3} p_{2_\delta} \kappa u_e^+ + \frac{1}{p_{2_\delta}} k_1 - \frac{1}{3 p_{2_\delta}^2} k_2) \right.$$

$$+ \Theta \left[\frac{1}{3} \kappa u_e^+ (p_{2_\delta} \kappa u_e^+ - 4 p_{3_\delta}) + \frac{2}{5 p_{2_\delta}} (\kappa u_e^+ p_{5_\delta} - 4/3\, \overline{p}_5) \right.$$

$$\left.\left. - \frac{2}{105} \frac{1}{p_{2_\delta}^2} (\kappa u_e^+ p_7^* - \frac{4}{3} \overline{p_7^*}) \right] \right\}$$

$$+ \frac{(\beta - \sigma - \Theta)}{\kappa^2 \alpha_s} \left\{ - \frac{1}{15} \kappa^2 u_e^{+2} p_{2_\delta} - \frac{4 \kappa u_e^+}{15 p_{2_\delta}} (2 p_{5_\delta} - 1) \right.$$

$$+ \frac{1}{p_{2_\delta}^2} \left[\frac{10}{3} n_2 + 2 L_3 - 2 n_1 + \frac{8}{315} \kappa u_e^+ (p_{7_\delta}^* + 2) \right]$$

$$- \frac{1}{3 p_{2_\delta}^3} \left[7 n_3 + 2 L_5 - 4 L_3 + 2 n_1 - \frac{2}{5} (3 m_6 - 5 m_4 \right.$$

$$\left. + 6 L_5 - 10 L_3) \right] + \frac{1}{15 p_{2_\delta}^4} \left[7 n_4 - \frac{2}{21} (15 m_8 \right.$$

$$\left.\left. - 42 m_6 + 35 m_4 + 30 L_7 - 84 L_5 + 70 L_3) \right] \right\}$$

$$G_{20} = \int_0^{\delta^+} u^+ \frac{\partial w^+}{\partial \beta} dy^+ = \frac{1}{\kappa^2 \alpha_s} \left[\frac{2}{p_{2_\delta}} M_1 - \frac{1}{p_{2_\delta}^2} M_2 \right]$$

$$G_{21} = \int_0^{\delta^+} (w^+ \frac{\partial w^+}{\partial \beta} - \frac{\partial w^+}{\partial y^+} J_5)\, dy^+ = \frac{1}{\kappa^2 \alpha_s p_{2_\delta}} \left\{ \sigma \kappa u_e^+ (2 K_1 - \frac{1}{p_{2_\delta}} K_2) \right.$$

$$+ \Theta \left[\frac{2}{3} p_{2_\delta}^2 \kappa^2 u_e^{+2} - 3 n_1 + \frac{2}{15} \bar{p}_5 + \frac{1}{p_{2_\delta}} (n_2 - \frac{2}{315} \bar{p}_7^*) \right]$$

$$+ \frac{(\beta-\sigma-\Theta)}{p_{2_\delta}} \left[\frac{2}{5} \kappa^2 u_e^{+2} p_{2_\delta}^3 + \frac{4}{15} p_{2_\delta} \kappa u_e^+ (p_{5_\delta} - 1) \right.$$

$$- \frac{10}{3} n_2 - 2 L_3 + 2 n_1 - \frac{4}{315} \kappa u_e^+ (p_{7_\delta}^* + 4)$$

$$+ \frac{1}{15 p_{2_\delta}} (35 n_3 - 2 L_5 + 10 n_1 - 6 m_6 + 10 m_4)$$

$$- \frac{1}{15 p_{2_\delta}^2} (7 n_4 - \frac{2}{21} (15 m_8 - 42 m_6 + 35 m_4$$

$$\left. \left. + 30 L_7 - 84 L_5 + 70 L_3) \right] \right\}$$

8. REFERENCES

1. White, F. M.; Lessmann, R. C.; and Christoph, G. H.: A Simplified Approach to the Analysis of Turbulent Boundary Layers in Two and Three Dimensions. NTIS AD-757 871, November 1972.

2. Jacobson, I. D.; Vollmer, A. G.; and Morton, J. B.: Calculation of the Velocity Profiles of the Incompressible Laminar Boundary Layer on a Yawed Spinning Cone and the Magnus Effect. BRL CR 110, July 1973.

3. White, F. M.; and Lessmann, R. C.: A Method of Direct Calculation of Compressible Turbulent Skin Friction and Heat Transfer in Three-Dimensional Boundary Layer Flow. AFFDL-TR-73-86, June 1973.

4. Sturek, W. B.: Three-Dimensional Boundary Layer Research as Applied to the Magnus Effect on Spinning Projectiles. BRL MR 2573, December 1975.

5. Sturek, W. B.: Boundary Layer Studies on a Spinning Cone. AIAA Paper No. 72-967, September 1972.

6. Sturek, W. B.: Boundary Layer Studies on a Spinning Cone. BRL R 1649, August 1973.

7. Jacobson, I. D.; and Morton, J. B.: Final Report on BRL Contract DAAD05-72-C-0131. BRL R ESS-3318-106-73U, January 1973.

8. Sturek, W. B.: Boundary-Layer Studies on Spinning Bodies of Revolution. BRL MR 2381, May 1974.

9. Potter, J. L.: Boundary-Layer Transition on Cones Near Mach One in an Aeroballistic Range. AEDC-TR-74-115, January 1975.

10. Schlichting, H.: Boundary-Layer Theory. McGraw-Hill, Inc., 1968.

11. Spalding, D. B.; and Chi, S. W.: The Drag of a Compressible Turbulent Boundary Layer on a Smooth Flat Plate With and Without Heat Transfer. J. Fluid Mech., Vol. 18, pt. 1, January 1964, pp. 117-143.

12. Isaacson, E.; and Keller, H. B.: Analysis of Numerical Methods. John Wiley and Sons, Inc., New York, 1966.

Figure 1 Coordinate System

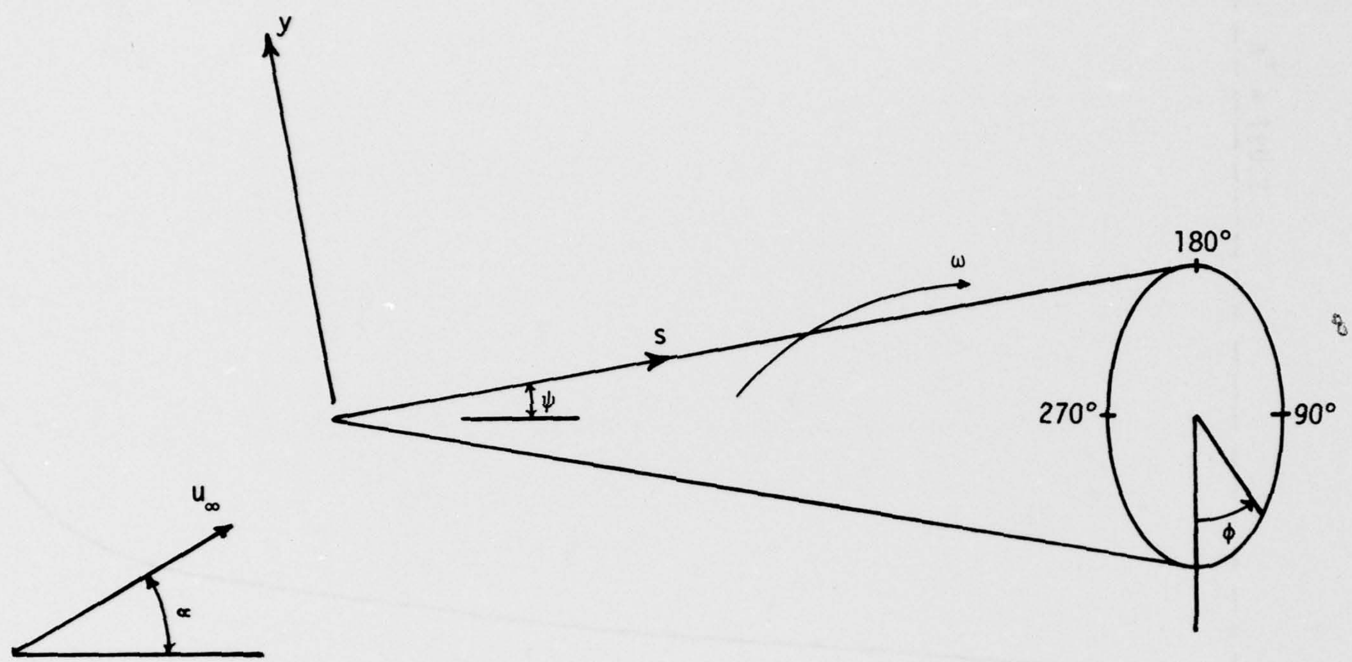


Figure 2 U^+ Velocity Profile

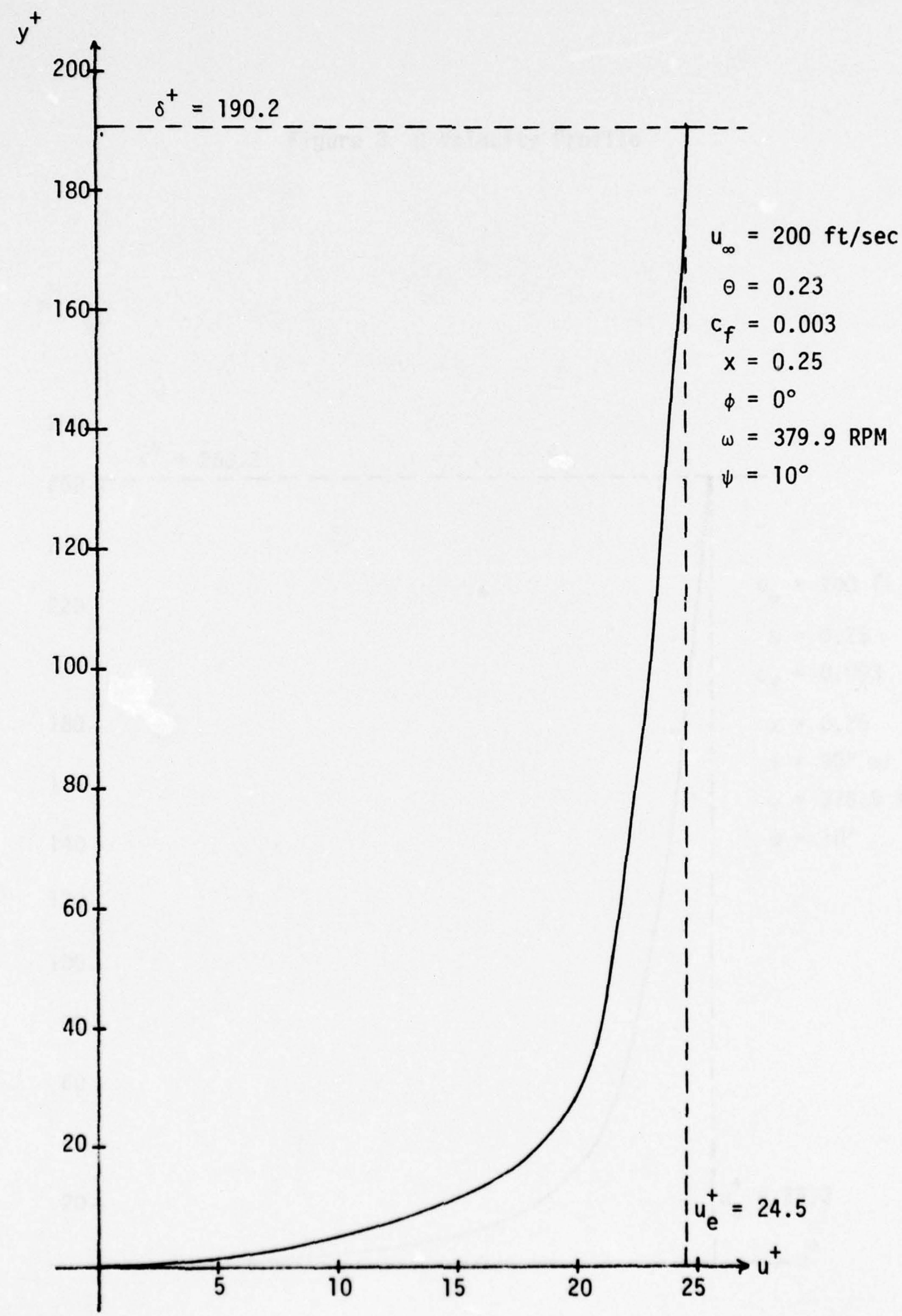


Figure 3 U Velocity Profile

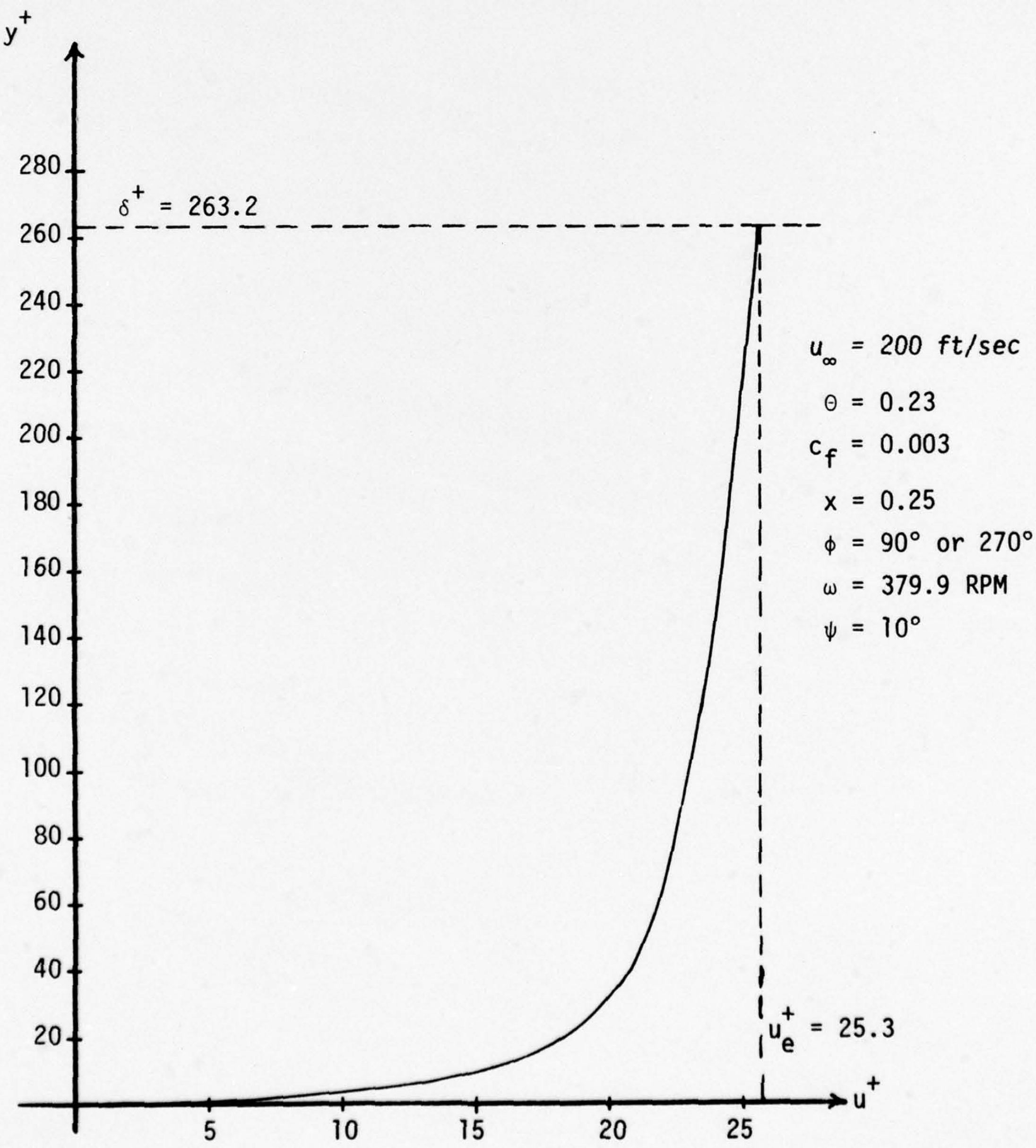


Figure 4 U Velocity Profile

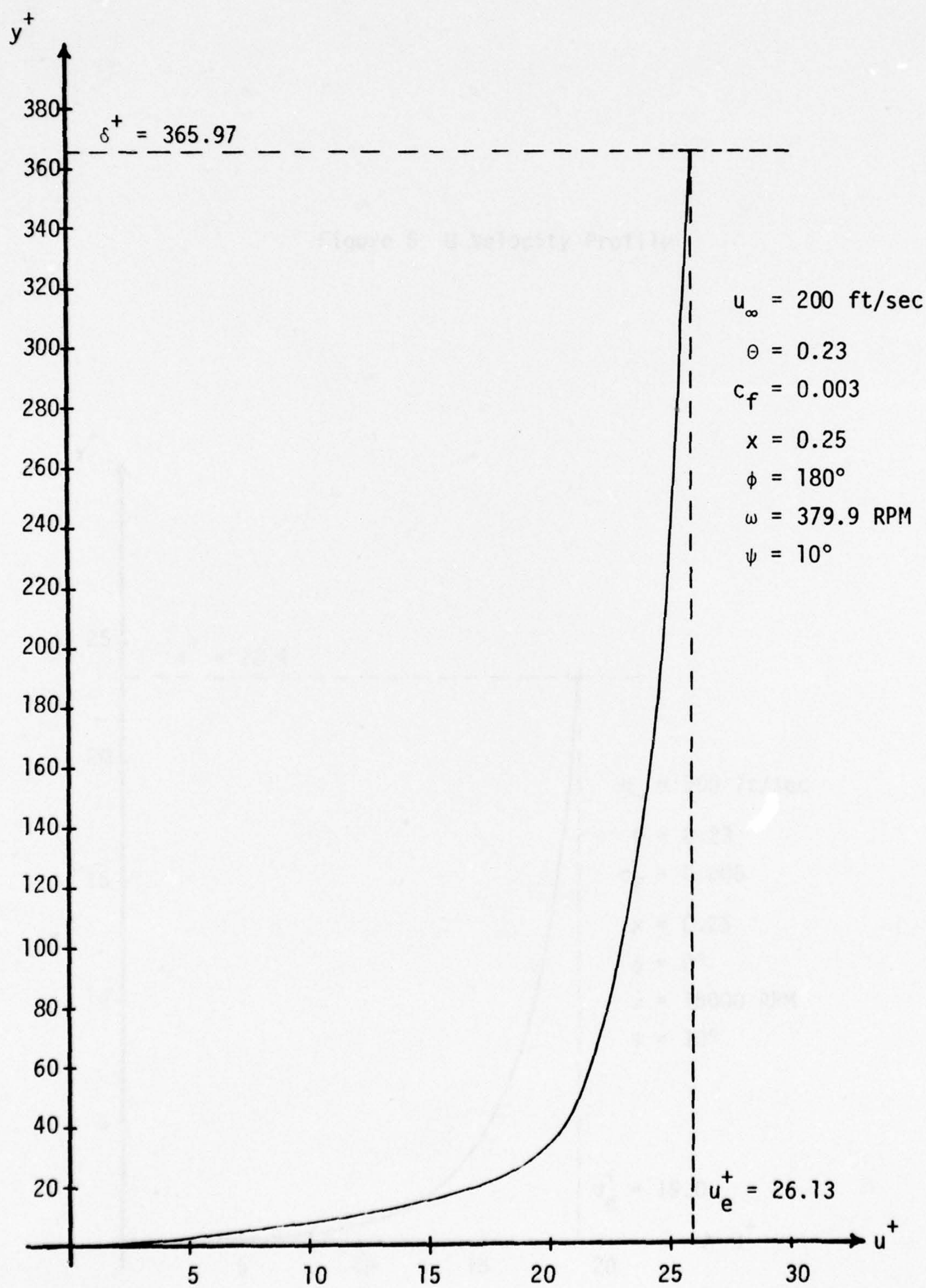


Figure 5 U Velocity Profile

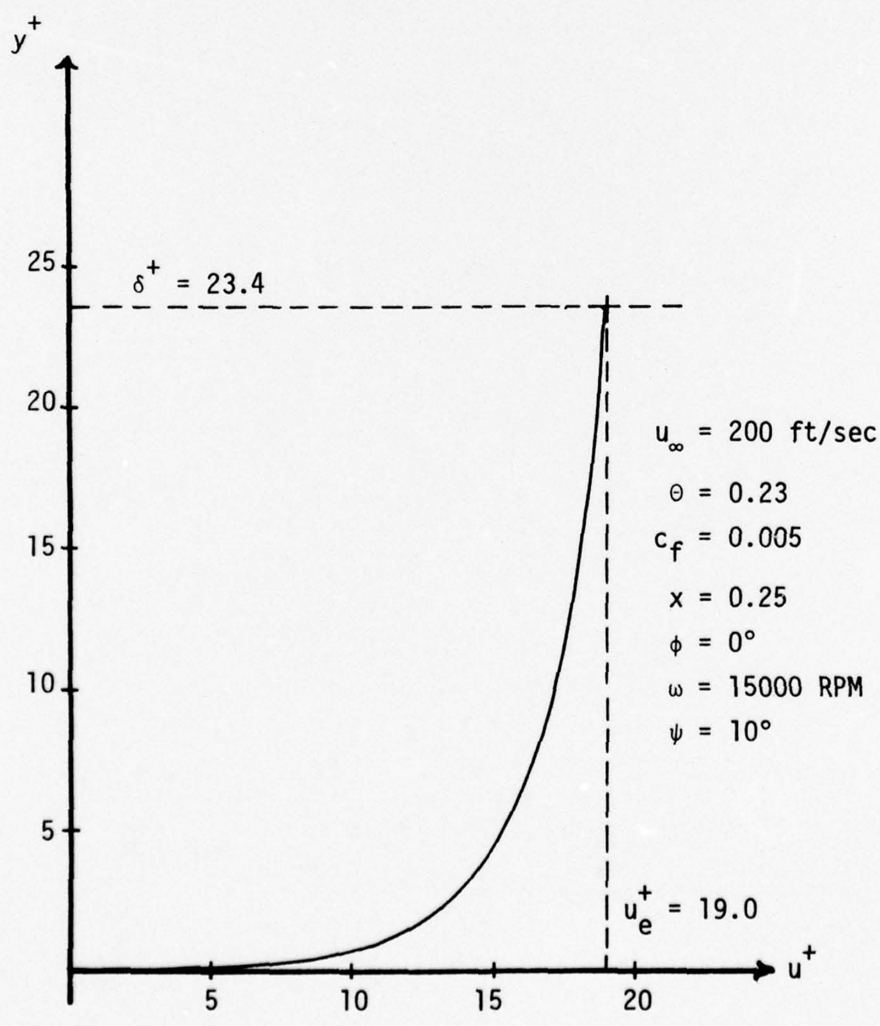


Figure 6 U Velocity Profile

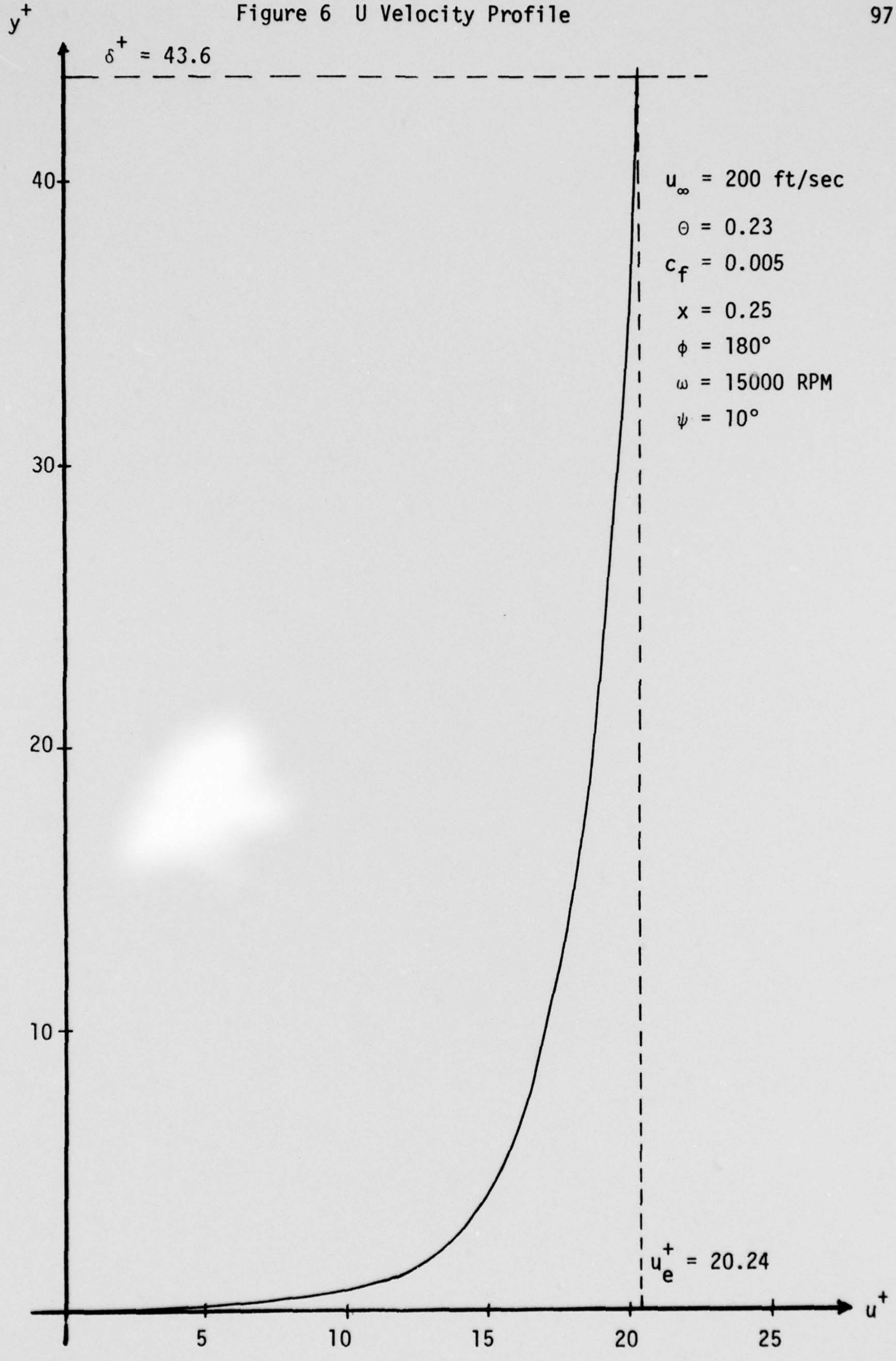


Figure 7 U Velocity Profile

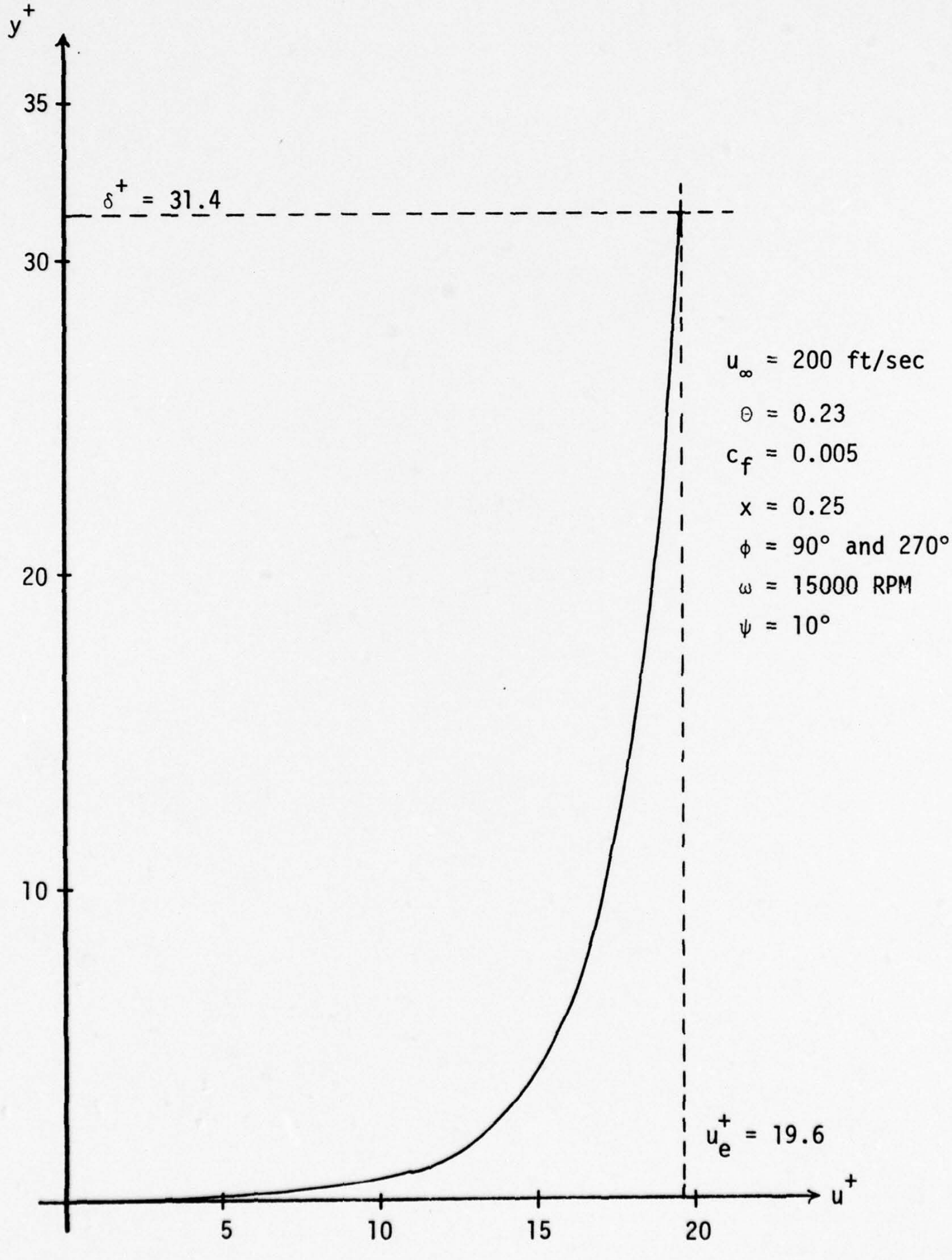


Figure 8 W Velocity Profile

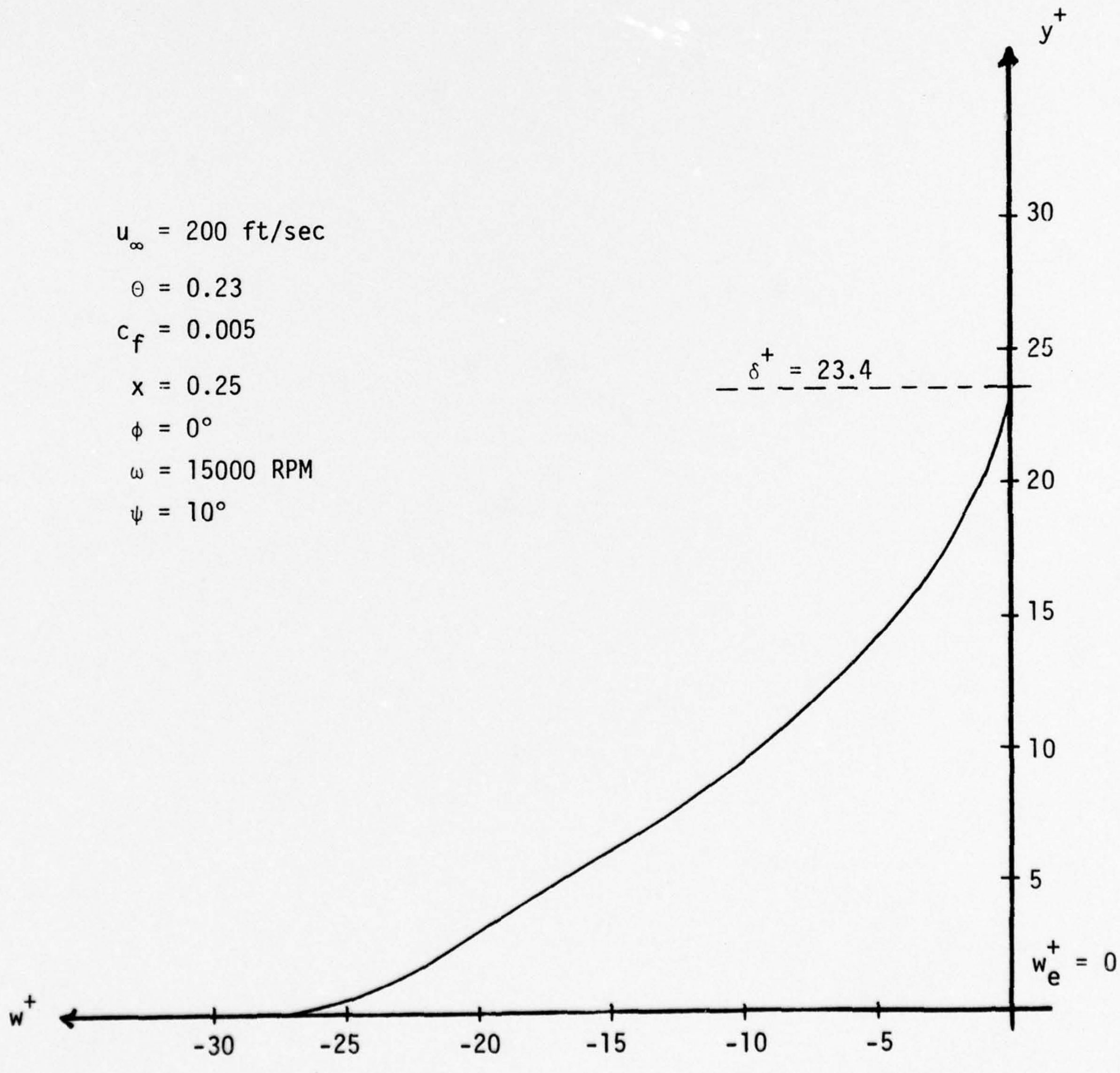


Figure 9 W Velocity Profile

u_∞ = 200 ft/sec

Θ = 0.23

c_f = 0.005

x = 0.25

ϕ = 90°

ω = 15000 RPM

ψ = 10°

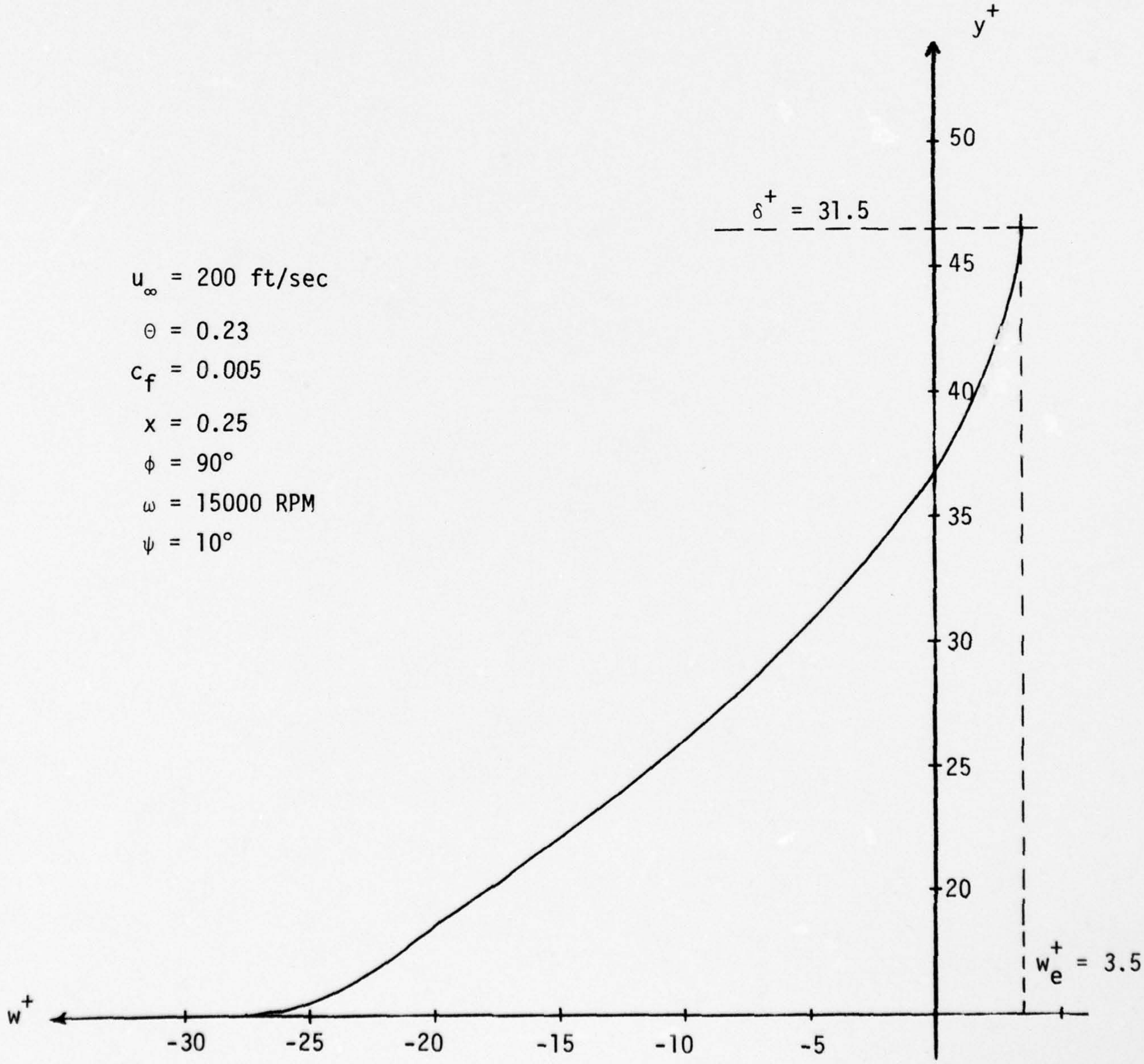


Figure 10 W Velocity Profile

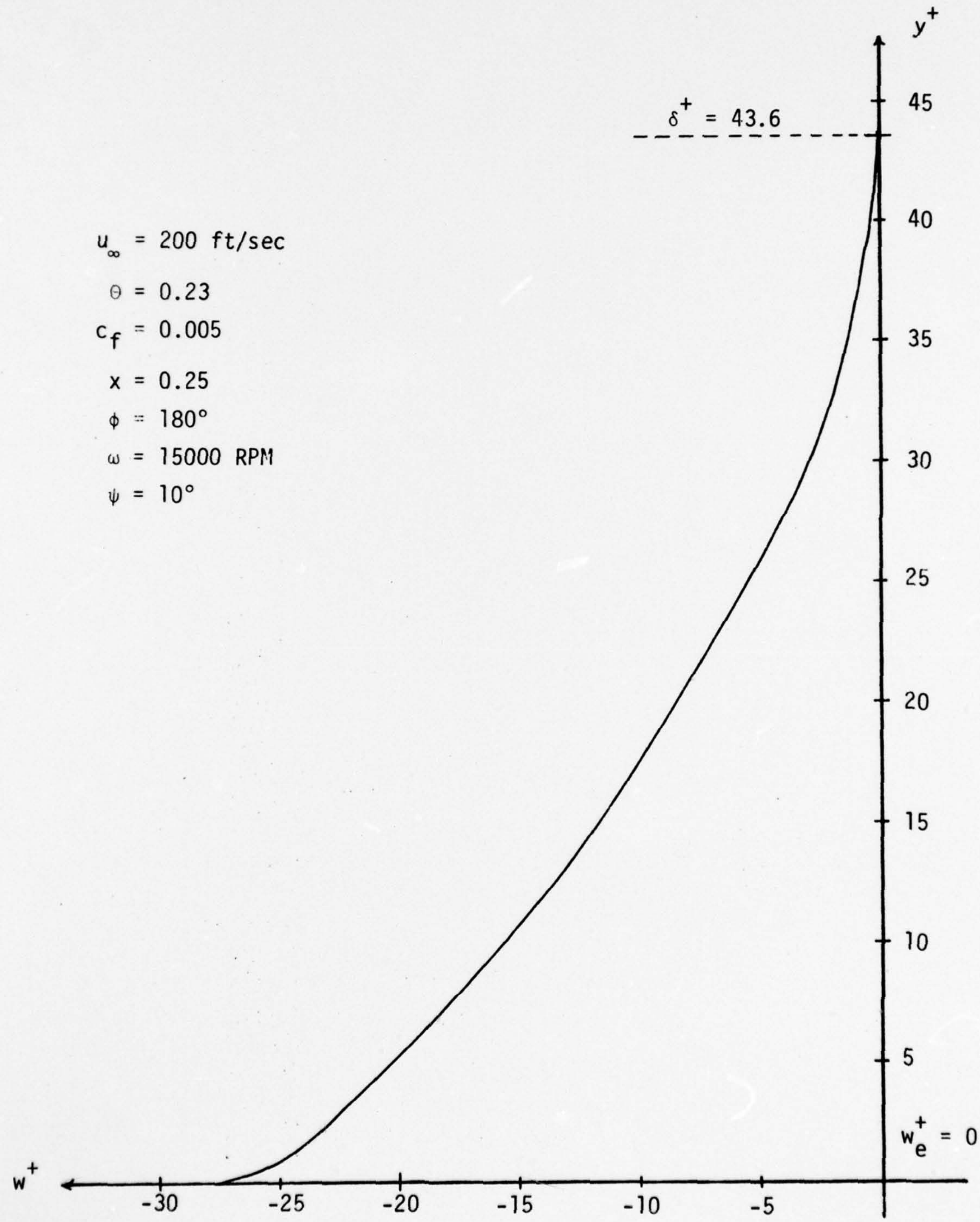


Figure 11 W Velocity Profile

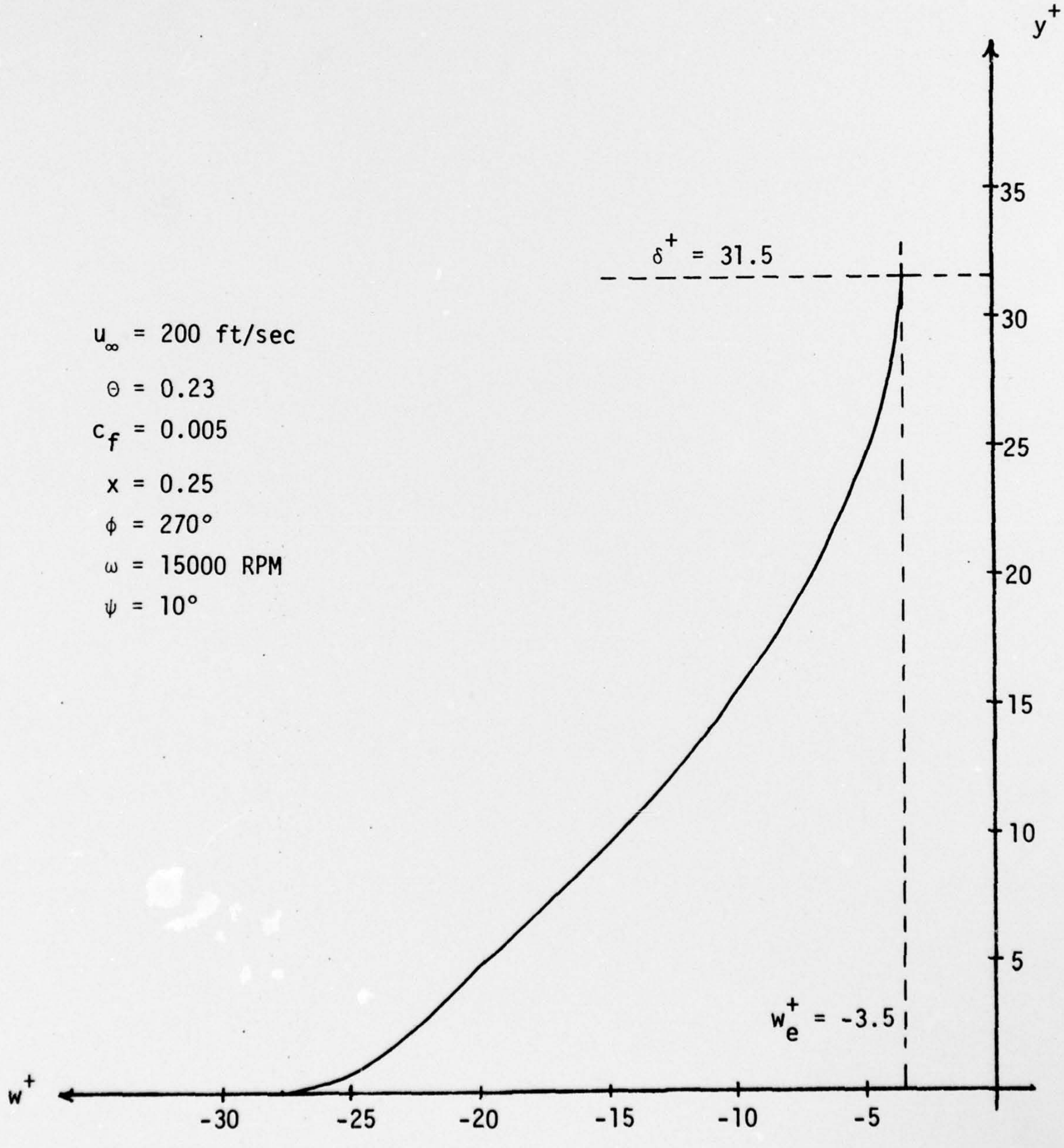
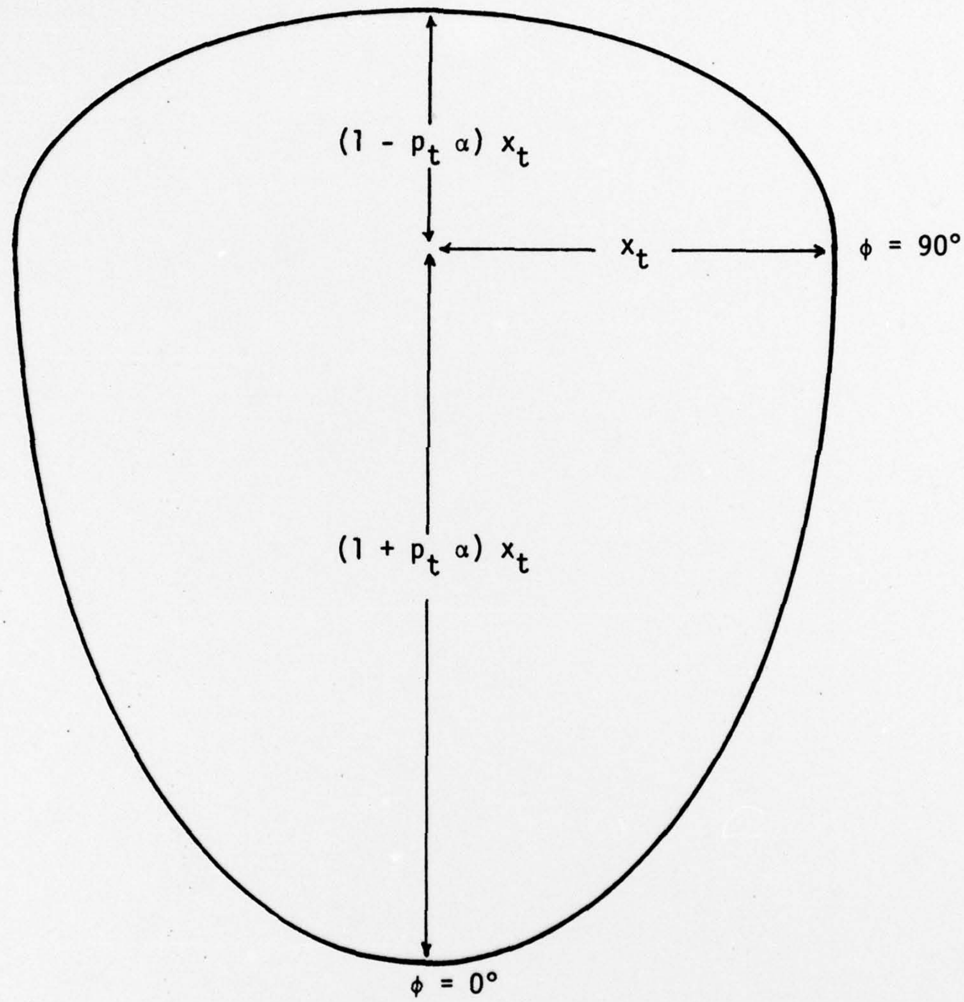


Figure 12 Transition Line



p_t = 5.7295

Figure 13 Grid for Finite Difference

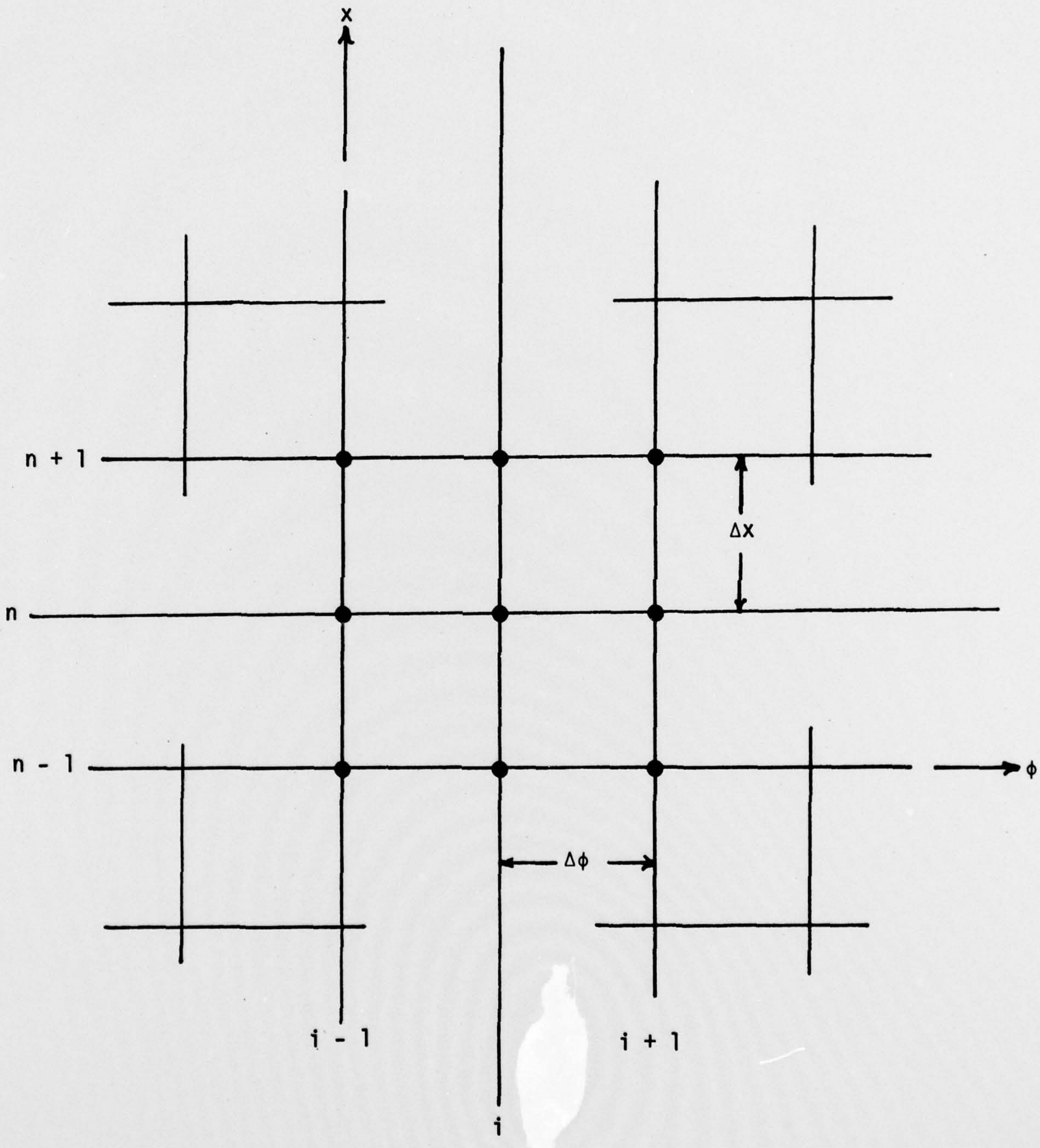


Figure 14 Boundary Layer Thickness

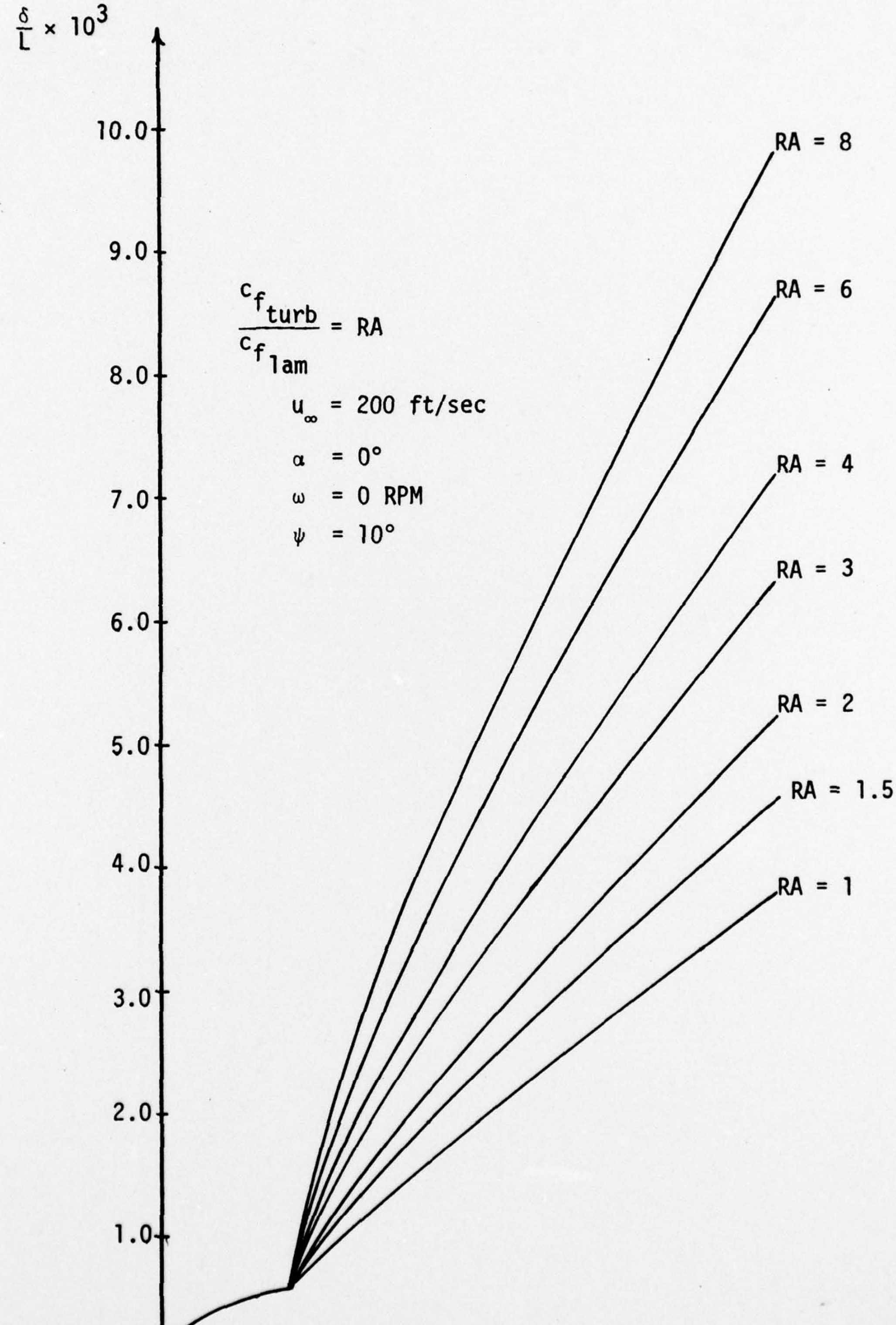


Figure 15 Turbulent Skin Friction

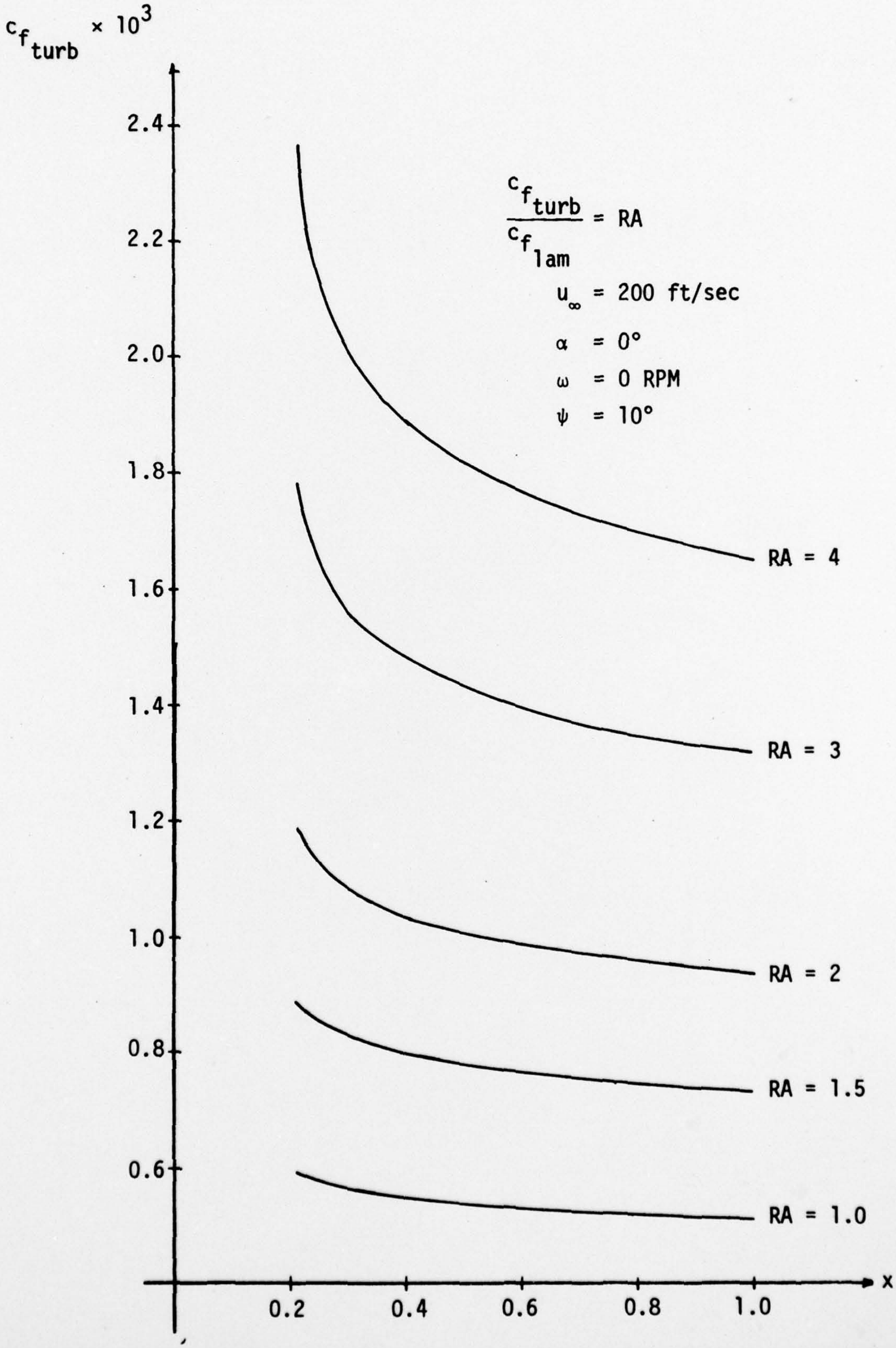


Figure 16 Boundary Layer Thickness

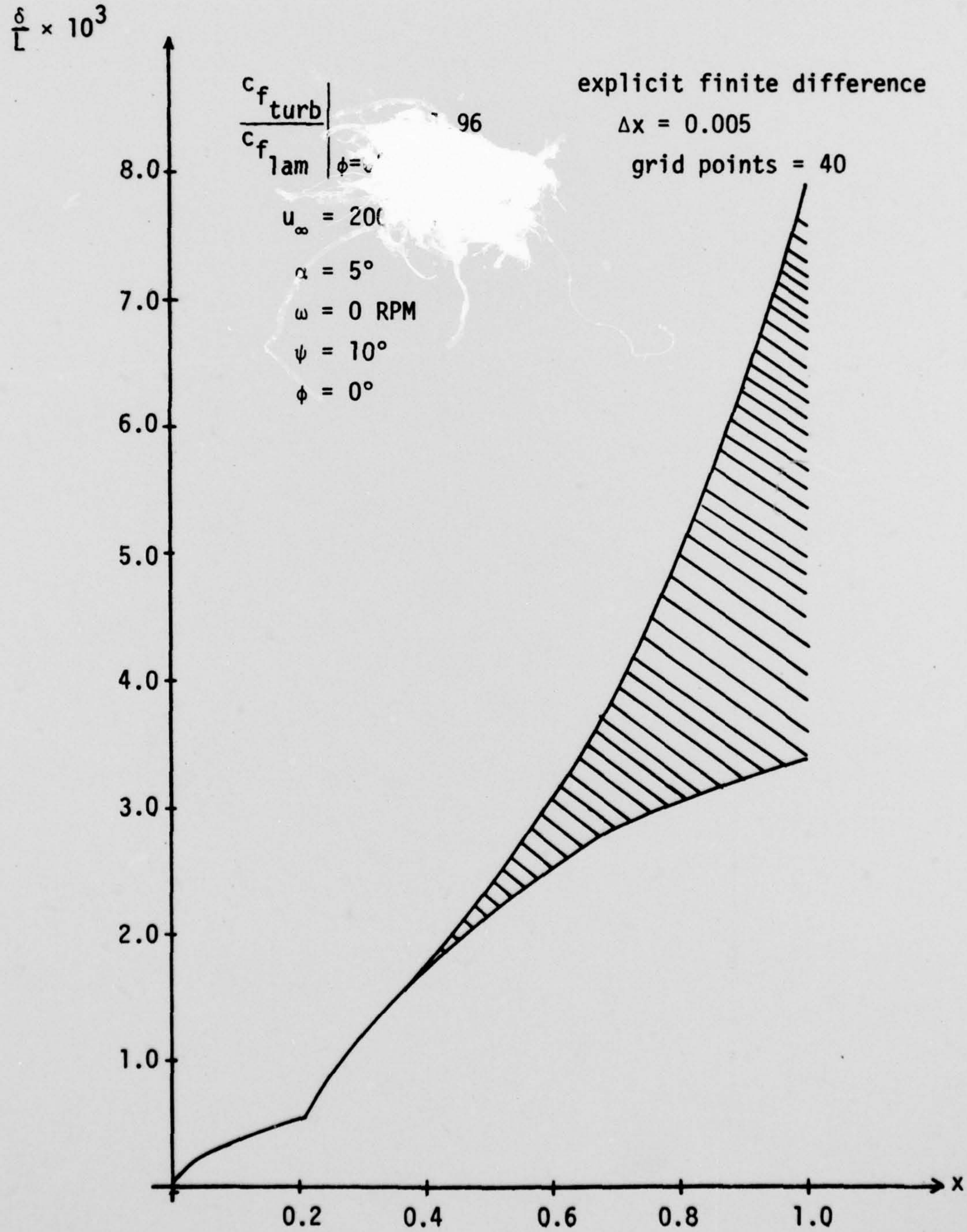


Figure 17 Boundary Layer Thickness

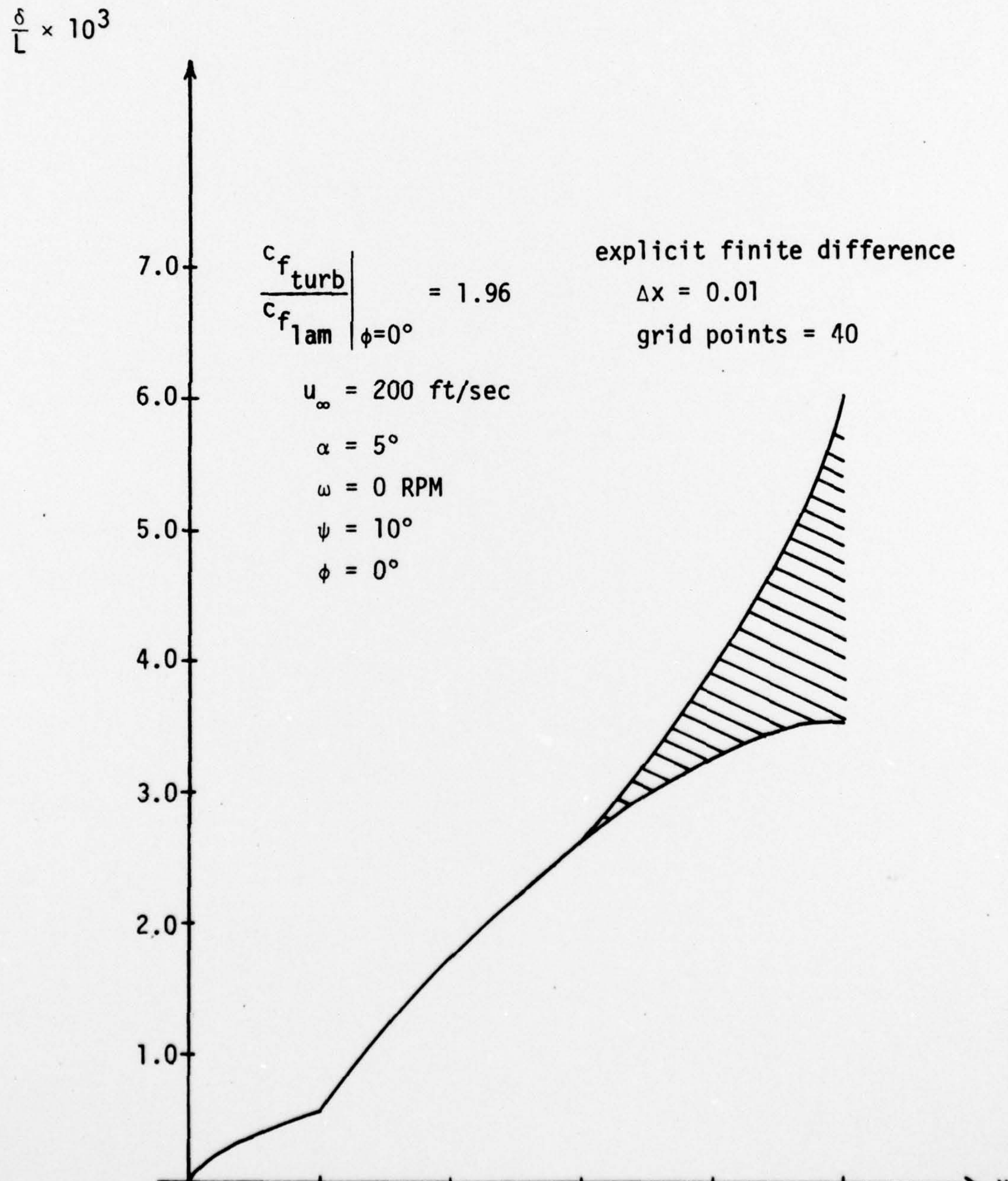


Figure 18 Boundary Layer Thickness

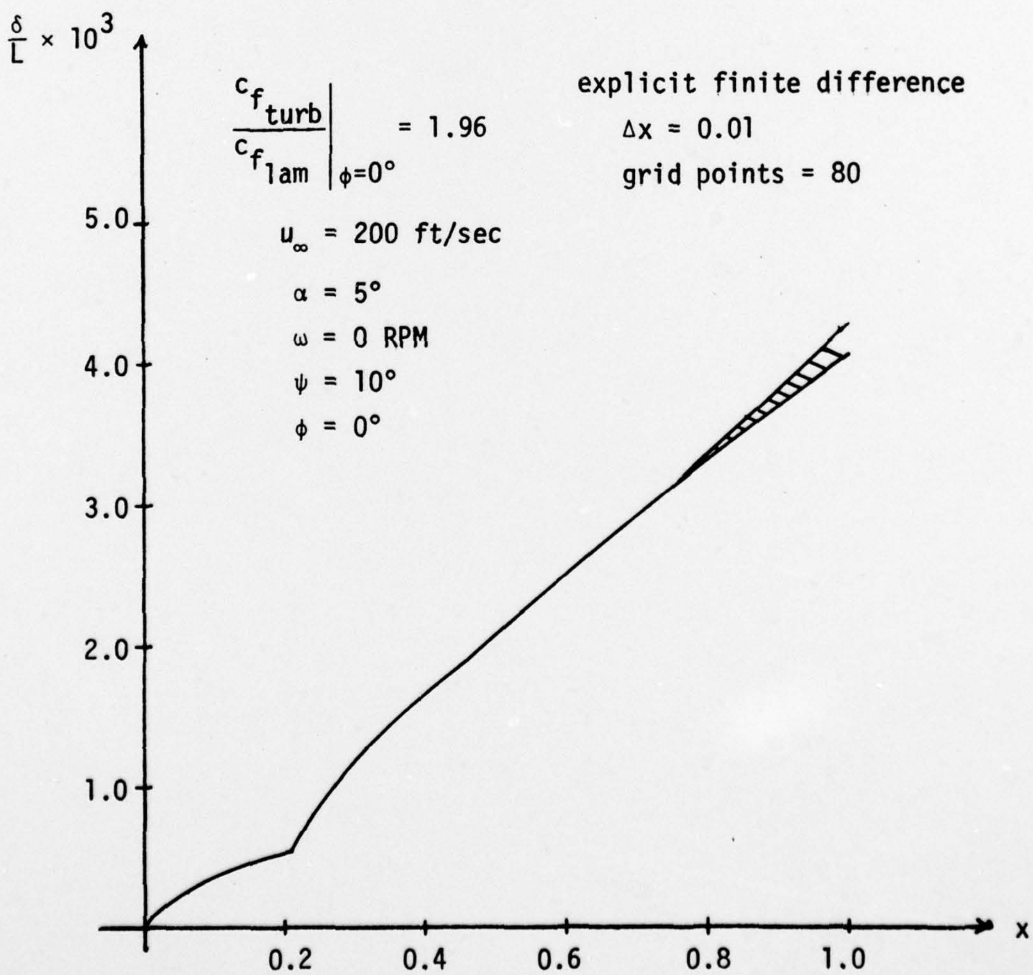


Figure 19 Boundary Layer Thickness

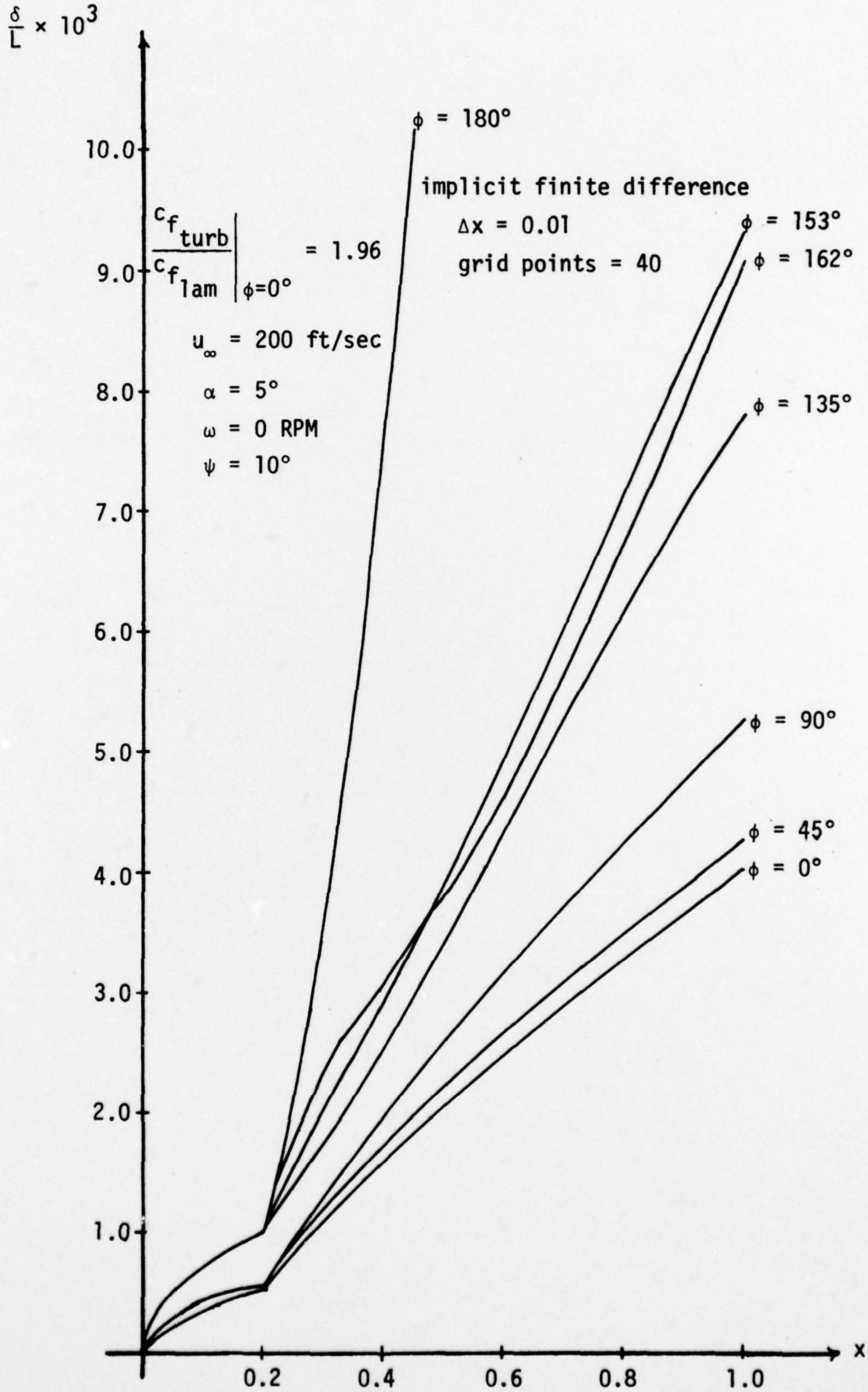
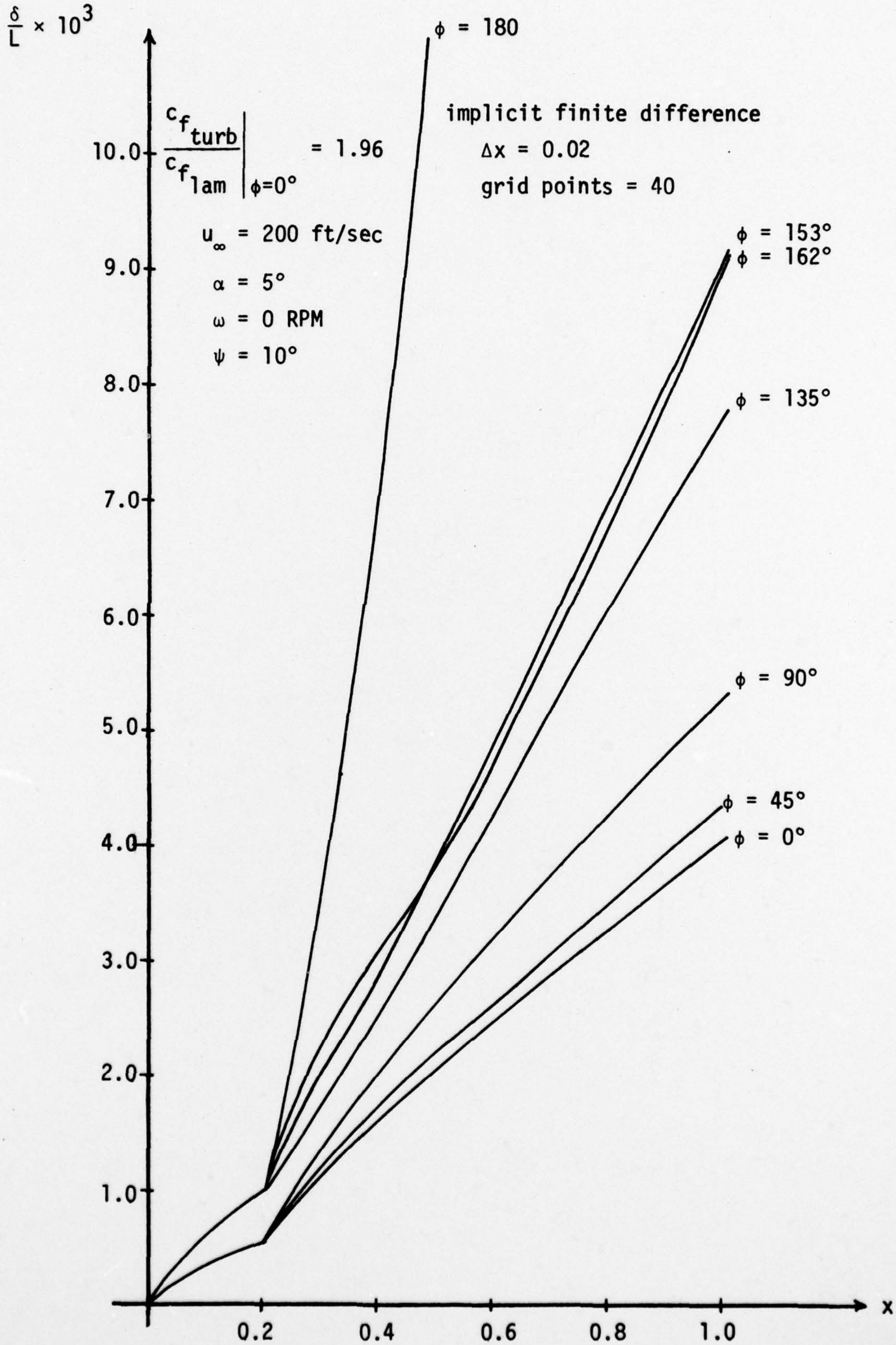
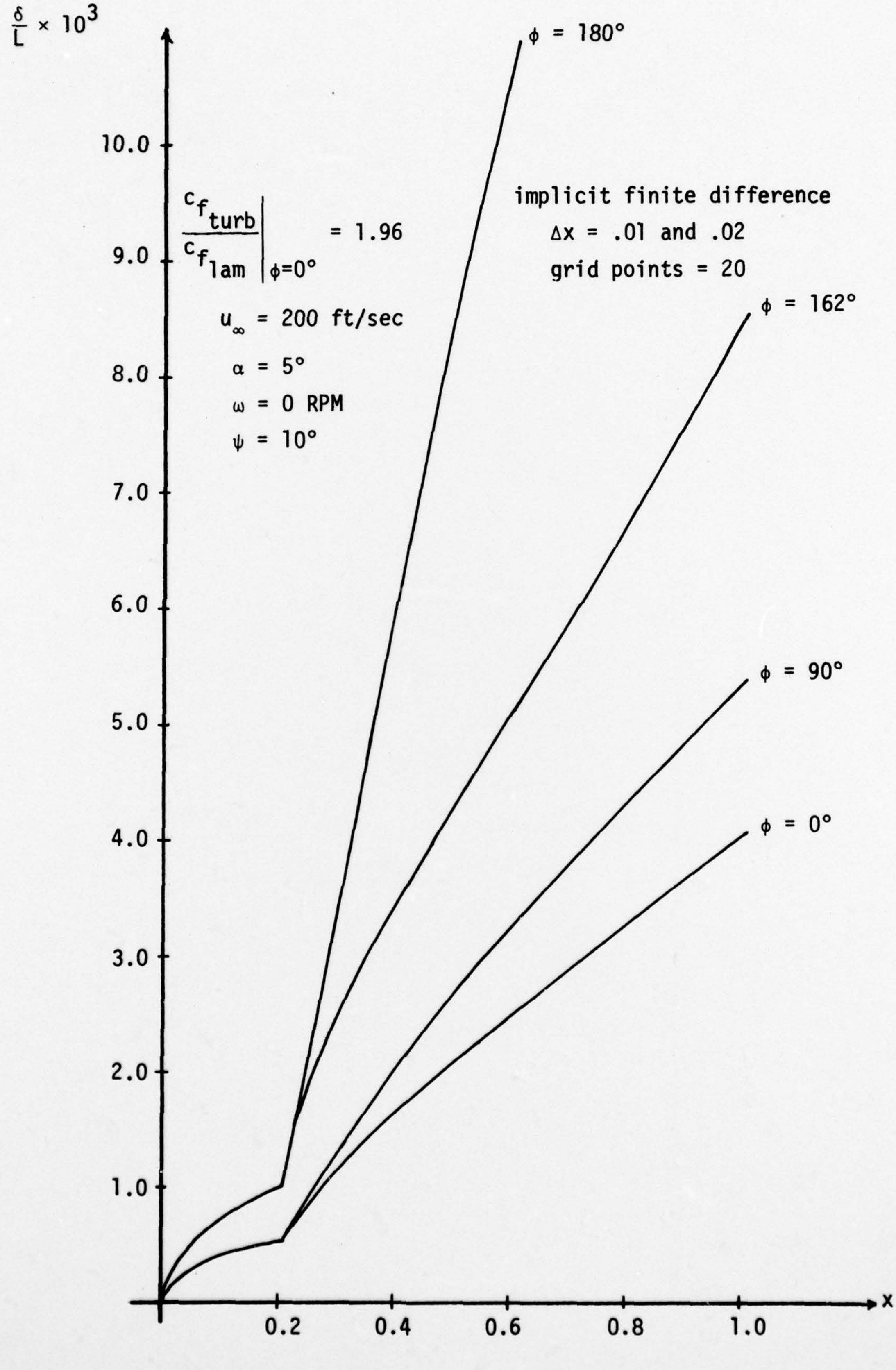


Figure 20 Boundary Layer Thickness





$\frac{\delta}{L} \times 10^3$
10.0
9.0
8.0
7.0
6.0
5.0
4.0
3.0
2.0
1.0
0.2
0.4
0.6
0.8
1.0
x
$\left.\frac{c_{f_{turb}}}{c_{f_{lam}}}\right|_{\phi=0°} = 1.96$
u_∞ = 200 ft/sec
α = 5°
ω = 0 RPM
ψ = 10°
implicit finite difference
Δx = .01 and .02
grid points = 20
ϕ = 180°
ϕ = 162°
ϕ = 90°
ϕ = 0°

Figure 22 Boundary Layer Thickness and Shear Angle

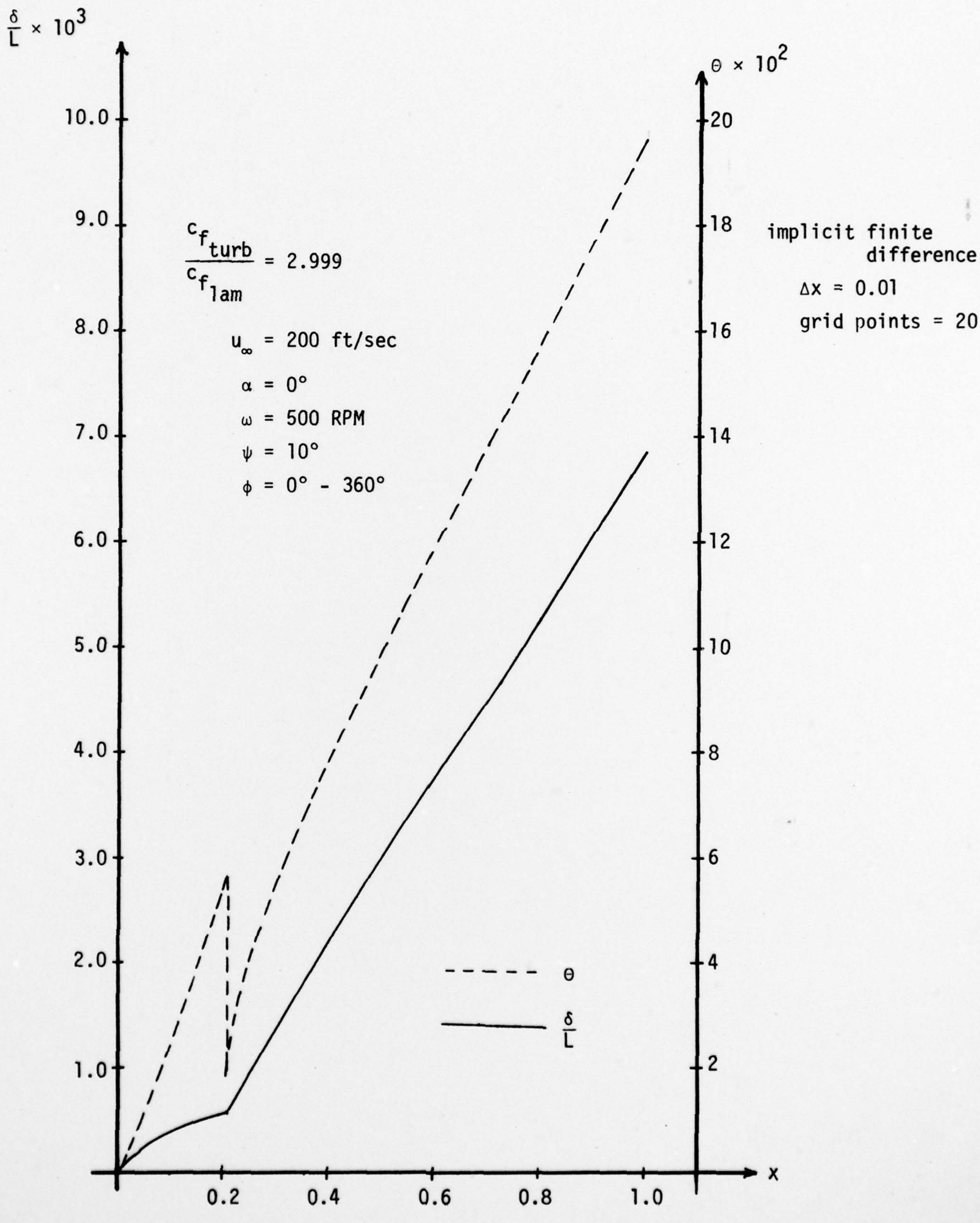


Figure 23 Boundary Layer Thickness and Shear Angle

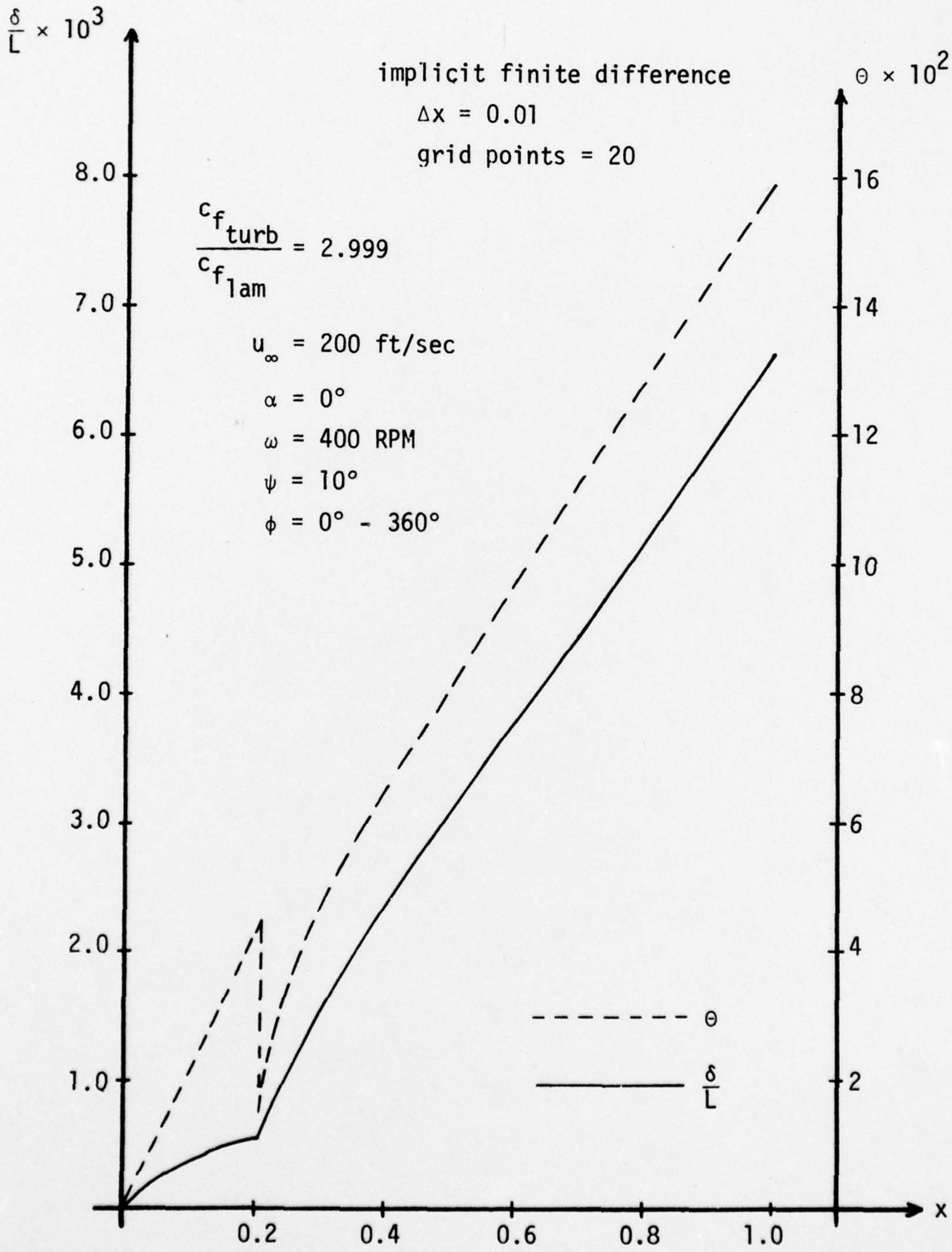


Figure 24 Boundary Layer Thickness

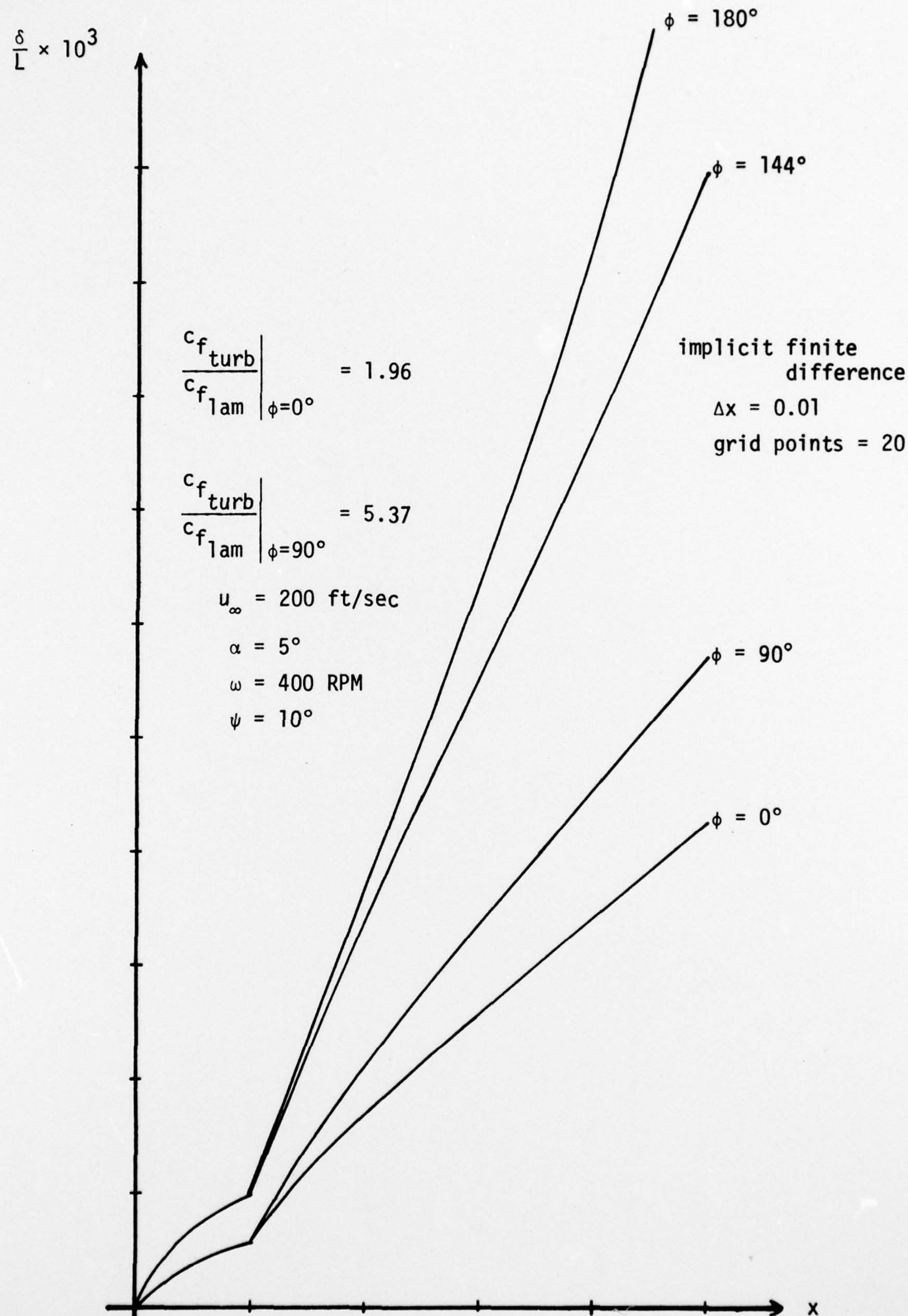


Figure 25 Skin Friction Shear Angle